

# Open Research Online

---

The Open University's repository of research publications and other research outputs

## On The Möbius Function Of Permutations Under The Pattern Containment Order

### Thesis

#### How to cite:

Marchant, David William (2020). On The Möbius Function Of Permutations Under The Pattern Containment Order. PhD thesis The Open University.

For guidance on citations see [FAQs](#).

© 2020 The Author



<https://creativecommons.org/licenses/by-nc-nd/4.0/>

Version: Version of Record

Link(s) to article on publisher's website:

<http://dx.doi.org/doi:10.21954/ou.ro.00011477>

---

Copyright and Moral Rights for the articles on this site are retained by the individual authors and/or other copyright owners. For more information on Open Research Online's data [policy](#) on reuse of materials please consult the policies page.

---

[oro.open.ac.uk](http://oro.open.ac.uk)

# ON THE MÖBIUS FUNCTION OF PERMUTATIONS UNDER THE PATTERN CONTAINMENT ORDER

DAVID WILLIAM MARCHANT



A thesis submitted to The Open University  
for the degree of Doctor of Philosophy  
in Mathematics

April 2020



# Abstract

We study several aspects of the Möbius function,  $\mu[\sigma, \pi]$ , on the poset of permutations under the pattern containment order.

First, we consider cases where the lower bound of the poset is indecomposable. We show that  $\mu[\sigma, \pi]$  can be computed by considering just the indecomposable permutations contained in the upper bound. We apply this to the case where the upper bound is an increasing oscillation, and give a method for computing the value of the Möbius function that only involves evaluating simple inequalities.

We then consider conditions on an interval which guarantee that the value of the Möbius function is zero. In particular, we show that if a permutation  $\pi$  contains two intervals of length 2, which are not order-isomorphic to one another, then  $\mu[1, \pi] = 0$ . This allows us to prove that the proportion of permutations of length  $n$  with principal Möbius function equal to zero is asymptotically bounded below by  $(1 - 1/e)^2 \geq 0.3995$ . This is the first result determining the value of  $\mu[1, \pi]$  for an asymptotically positive proportion of permutations  $\pi$ .

Following this, we use “2413-balloon” permutations to show that the growth of the principal Möbius function on the permutation poset is exponential. This improves on previous work, which has shown that the growth is at least polynomial.

We then generalise 2413-balloon permutations, and find a recursion for the value of the principal Möbius function of these generalisations.

Finally, we look back at the results found, and discuss ways to relate the results from each chapter. We then consider further research avenues.



# Dedication

This thesis is dedicated to Jo. Five years ago she agreed that I could stop work in order to study for a PhD – which has been the hardest I have ever worked. There is no way for me to adequately express my gratitude for her support, her tolerance for my mathematical adventures, or the way that she makes me complete.

אני לדודי ודודי לי  
שר השירים ו:ג



# Acknowledgements

I would like to thank my supervisor, Robert Brignall, for introducing me to the Möbius function on the permutation pattern poset, and for offering me the chance to study under his supervision. Robert has always been patient and forbearing in our interactions, and has allowed me the freedom to find my own mathematical “voice”.

As a part-time student, living some distance from The Open University campus, my opportunities to meet with other PhD students have been somewhat limited. Where these opportunities have arisen, I have been warmly welcomed. I would especially like to thank Grahame Erskine, Jakub Sliáčan, James Tuite, Margaret Stanier, Olivia Jeans, and Rob Lewis for their support and discussions.

Part-time students are not entitled to any funding support from the School of Mathematics and Statistics at The Open University. I am therefore most grateful to the School for providing funding for conferences and visits over the last four years. I am also grateful for the support received from the National Science Foundation for contributions towards travel and accommodation costs for attending conferences, and for the support from Charles University in Prague for my visit in 2019.

The Permutation Patterns community is small and vibrant. I was made to feel welcome by everyone I encountered. I would particularly like to thank Einar Steingrímsson, Vít Jelínek, and Jan Kynčl for their support and encouragement. I would also like to thank the anonymous referees of the papers which underpin this thesis for all their work.





# Declarations

Some chapters of this thesis are based on work that has been published. The relevant chapters are as follows:

1. Chapter [4](#) is based on published joint work with Robert Brignall. The paper [\[16\]](#) was published in *Discrete Mathematics*.
2. Chapter [5](#) is based on published joint work with Robert Brignall, Vít Jelínek and Jan Kynčl. The paper [\[15\]](#) was published in *Mathematika*. I thank the London Mathematical Society for granting permission to include edited extracts from the published article in this thesis.
3. Chapter [6](#) is based on published sole work by the author. The paper [\[31\]](#) was published in *The Electronic Journal of Combinatorics*.

None of the results appear in any other thesis, and all co-authors have agreed with the inclusion of joint work in this thesis. Where work has been published, the publishers have given permission for edited extracts of the published article to be included in this thesis.

This thesis is approximately 49,000 words.



# Contents

<b>Abstract</b>	<b>iii</b>
<b>Dedication</b>	<b>v</b>
<b>Acknowledgements</b>	<b>vii</b>
<b>Declarations</b>	<b>ix</b>
<b>1 Overview</b>	<b>1</b>
1.1 Introductory material . . . . .	1
1.2 Chapters based on published material . . . . .	1
1.3 Conclusion . . . . .	2
1.4 Details of chapters based on published material . . . . .	2
1.4.1 The Möbius function of permutations with an indecomposable lower bound . . . . .	2
1.4.2 Zeros of the Möbius function of permutations . . . . .	3
1.4.3 2413-balloons and the growth of the Möbius function . . . . .	5
1.4.4 The principal Möbius function of balloon permutations . . . . .	6

<b>2</b>	<b>Common definitions</b>	<b>9</b>
<b>3</b>	<b>Background and history</b>	<b>17</b>
3.1	Permutations . . . . .	17
3.2	The Möbius function . . . . .	19
3.2.1	Simplicial complexes . . . . .	19
3.2.2	Incidence algebras and incidence matrices . . . . .	20
3.3	The Möbius function for general posets . . . . .	22
3.4	The Möbius function of the permutation poset under pattern containment	25
3.5	Motivation . . . . .	30
<b>4</b>	<b>The Möbius function of permutations with an indecomposable lower bound</b>	<b>33</b>
4.1	Preamble . . . . .	33
4.2	Introduction . . . . .	34
4.3	Definitions and notation . . . . .	36
4.4	Preliminary lemmas and main theorem . . . . .	39
4.5	Increasing oscillations . . . . .	45
4.5.1	Example of Theorem 23 . . . . .	54
4.6	Concluding remarks . . . . .	57
4.7	Chapter summary . . . . .	60

<b>5</b>	<b>Zeros of the Möbius function of permutations</b>	<b>63</b>
5.1	Preamble . . . . .	63
5.2	Introduction . . . . .	64
5.3	Definitions and notation . . . . .	66
5.3.1	Intervals with vanishing Möbius function . . . . .	68
5.3.2	Embeddings . . . . .	69
5.3.3	Möbius function via normal embeddings . . . . .	71
5.4	Permutations with opposing adjacencies . . . . .	75
5.5	A general construction of $\sigma$ -annihilators . . . . .	78
5.6	The density of zeros . . . . .	81
5.7	More complicated examples . . . . .	82
5.8	Concluding remarks . . . . .	88
5.9	Chapter summary . . . . .	90
5.9.1	Improving the lower bound for the density of zeros, $d_n$ . . . . .	90
5.9.2	Extending the “opposing adjacencies” theorem . . . . .	93
<b>6</b>	<b>2413-balloons and the growth of the Möbius function</b>	<b>97</b>
6.1	Preamble . . . . .	97
6.2	Introduction . . . . .	98
6.3	Essential definitions, notation, and results . . . . .	100
6.4	2413-Balloons . . . . .	102
6.5	The principal Möbius function of double 2413-balloons . . . . .	110
6.6	The growth of the Möbius function . . . . .	112

6.7	The principal Möbius function of 2413-balloons . . . . .	114
6.8	Concluding remarks . . . . .	118
6.8.1	Generalising the balloon operator . . . . .	118
6.8.2	Generalised 2413-balloons . . . . .	118
6.8.3	Bounding the Möbius function on hereditary classes . . . . .	119
6.9	Chapter summary . . . . .	121
<b>7</b>	<b>The principal Möbius function of balloon permutations</b>	<b>123</b>
7.1	Preamble . . . . .	123
7.2	Introduction . . . . .	124
7.3	Definitions, examples and notation . . . . .	125
7.3.1	Balloon permutations . . . . .	125
7.3.2	Types of balloon permutations . . . . .	126
7.3.3	Notation . . . . .	128
7.4	The principal Möbius function of balloon permutations . . . . .	133
7.5	The principal Möbius function of wedge permutations . . . . .	137
7.6	Chapter summary . . . . .	140
<b>8</b>	<b>Conclusion</b>	<b>141</b>
8.1	A review of our results . . . . .	141
8.2	Further research into the Möbius function . . . . .	143
8.2.1	The Möbius function of simple permutations . . . . .	143
8.3	Further research into global properties of the poset . . . . .	149

8.3.1	Extremal values of $\mu[\pi]$ as a function of the length of the permutations . . . . .	149
8.3.2	Characterising permutations where $\mu[\pi]$ achieves extreme values	149
<b>A</b>	<b>Values of <math>\mu[\pi]</math> where <math>\pi</math> is a specific simple permutation</b>	<b>153</b>
A.1	Values of $\mu[W_1(n)]$ . . . . .	153
A.2	Values of $\mu[W_2(n)]$ . . . . .	154
A.3	Values of $\mu[E_1(2n, k)]$ . . . . .	155
A.4	Values of $\mu[E_2(2n)]$ . . . . .	157
A.5	Values of $\mu[O(2n + 1, k)]$ . . . . .	158
<b>B</b>	<b>Canonical permutations that achieve <math>\text{Max}_\mu(n)</math> or <math>\text{Min}_\mu(n)</math></b>	<b>159</b>
	<b>Bibliography</b>	<b>161</b>
	<b>Index</b>	<b>167</b>





# Chapter 1

## Overview

This thesis is primarily concerned with the Möbius function on the poset of permutations ordered by classic pattern containment.

This thesis consists of eight chapters.

### 1.1 Introductory material

This chapter (Chapter 1), is an overview of the thesis. Following this overview, we have Chapter 2, which defines the terminology and notation for the subject area as a whole. Subsequent chapters will also include definitions of terminology and notation that is only used in those chapters. This is then followed, in Chapter 3, with a brief overview of the history of the subject area, and a description of the motivation for the work described in this thesis.

### 1.2 Chapters based on published material

Chapters 4, 5, and 6 are based on material that has been published in peer-reviewed journals. Chapter 7 is based on material currently being prepared for publication.

These chapters start with a section (“Preamble”) that introduces the subject matter. This is based on the abstract of the published paper, but may include additional material to help place the subject into the context of this thesis. This is then followed by sections that are based on the published material. We then conclude each chapter with a section (“Chapter summary”) that summarises the impact of the results, discusses how they relate to this thesis, and considers possible avenues for further research following on from the results described in the chapter.

## 1.3 Conclusion

Chapter 8 looks back at the results from chapters 4, 5, 6, and 7. Here we summarise the results that we presented in the preceding four chapters, and discuss whether it is possible to find a common theme (beyond the obvious “related to the Möbius function”) in the work presented. We then consider possible avenues for future research.

## 1.4 Details of chapters based on published material

We now provide a more detailed description of the chapters based on published material, or on material being prepared for publication.

In the description that follows, we may use terminology that is in common use in the field, but which will not be formally defined until Chapter 2.

### 1.4.1 The Möbius function of permutations with an indecomposable lower bound

Chapter 4 is based on a published paper, “The Möbius function of permutations with an indecomposable lower bound” [16], which is joint work with Robert Brignall.

In this paper we show that, given some interval  $[\sigma, \pi]$  in the permutation poset, if  $\sigma$  is sum (resp. skew) indecomposable, then the value of the Möbius function  $\mu[\sigma, \pi]$  depends solely on the sum (resp. skew) indecomposable permutations contained in the upper bound  $\pi$ .

The basic methodology is to first use existing results to show that certain permutations that are contained in the interval do not contribute to the value of the Möbius function. We then show that the permutations that remain can be partitioned into families, defined by a single sum (resp. skew) indecomposable permutation  $\alpha$ , and that the net contribution of a family will be in  $\{\pm\mu[\sigma, \alpha], 0\}$ . We derive a  $\{\pm 1, 0\}$  weighting function  $W(\sigma, \alpha, \pi)$ , and using this, we then show that  $\mu[\sigma, \pi]$  can be calculated by summing the value of  $\mu[\sigma, \alpha]W(\sigma, \alpha, \pi)$ , over all permutations  $\alpha$  that are sum (resp. skew) indecomposable and contained in the interval.

We then set  $\pi$  to be an increasing oscillation. This allows us to define a revised weighting function specific to these intervals which can be computed by using simple inequalities. This then leads to a fast algorithm for calculating  $\mu[\sigma, \pi]$ , where  $\pi$  is an increasing oscillation.

We then have some conjectures relating to the long-term behaviour of the absolute value of  $\mu[1, \pi]$ , where  $\pi$  is an increasing oscillation.

The chapter concludes by summarising the impact of the results, particularly from a computational perspective.

### 1.4.2 Zeros of the Möbius function of permutations

Chapter 5 is based on a published paper “Zeros of the Möbius function of permutations” [15], which is joint work with Robert Brignall, Vít Jelínek and Jan Kynčl.

In this paper we show that if a permutation  $\pi$  contains two opposing adjacencies, then  $\mu[1, \pi] = 0$ . We then use this result to show that the proportion of permutations of length  $n$  with principal Möbius function equal to zero is, asymptotically, bounded below by 0.3995.

We start by showing that if a poset  $P$  has a particular structure, then  $\mu[P] = 0$ . We then show that if a permutation  $\pi$  contains two opposing adjacencies, then the poset interval  $[1, \pi]$  has the required structure, and it follows that  $\mu[1, \pi] = 0$ . We then provide a second proof of the same result based on normal embeddings. The techniques used in both proofs are used in later, more complicated, settings.

We then show that if  $\sigma$  is any permutation, and  $\phi$  meets certain requirements, then any permutation  $\pi$  that contains an interval order-isomorphic to  $\phi$  has  $\mu[\sigma, \pi] = 0$ . We use this result to show if  $\sigma$  meets a particular condition, and  $\pi$  contains an interval copy in the form  $\alpha \oplus 1 \oplus \beta$ , then  $\mu[\sigma, \pi] = 0$ . We then show that if  $\sigma = 1$ , then  $\sigma$  meets the condition required, and thus we prove that if a permutation contains an interval copy in the form  $\alpha \oplus 1 \oplus \beta$ , then  $\mu[1, \pi] = 0$ .

In the next part of this chapter, we show that, asymptotically, the proportion of permutations of length  $n$  that have a principal Möbius function value of zero,  $d_n$ , is bounded below by  $\left(1 - \frac{1}{e}\right)^2 \cong 0.3995$ .

We then use the techniques already introduced to show that there are pairs of permutations,  $\alpha, \beta$ , such that if  $\pi$  contains interval copies of  $\alpha$  and  $\beta$ , then  $\mu[1, \pi] = 0$ . We further show that there are individual permutations  $\alpha$  with the property that if  $\pi$  contains an interval copies of  $\alpha$ , then  $\mu[1, \pi] = 0$ .

We then discuss further ways in which we could find a permutation  $\pi$  where the presence of a specific interval or intervals in  $\pi$  would guarantee  $\mu[1, \pi] = 0$ . We discuss  $d_n$ , including a conjecture on an upper bound for  $d_n$ .

The chapter concludes by summarising the impact of the results. We show that there is some numerical evidence that a large proportion of permutations with multiple non-opposing adjacencies have a principal Möbius function value of zero, and show that if we could prove this for a positive proportion of these permutations, then we could improve the lower bound of  $d_n$ .

We discuss extending the opposing adjacency result to more general poset intervals and show that this is not possible in all cases. We then present two minor results. The first shows that certain intervals  $[\sigma, \pi]$  have  $\mu[\sigma, \pi] = 0$ . The second result shows

that if  $\sigma$  is adjacency-free, and  $\pi$  is an inflation of  $\sigma$ , and  $\mu[\sigma, \pi] = 0$ , then we have some information about the permutations used in the inflation.

### 1.4.3 2413-balloons and the growth of the Möbius function

Chapter 6 is based on a published paper “2413-balloons and the growth of the Möbius function” [31], which is sole work by the author. In this paper we show that the growth of the principal Möbius function on the permutation poset is exponential.

We start by defining the “2413-balloon” of some permutation  $\beta$ . The resulting permutation has extremal points that are order-isomorphic to 2413, and the non-extremal points are an interval copy of  $\beta$ . A double 2413-balloon is the result of ballooning a permutation that is already a 2413-balloon.

We take a poset where the upper bound is a double 2413-balloon, and the lower bound is 1, and we show how we can partition the chains in the poset into three sets. We then show that two of these subsets contribute zero to the value of the Möbius function.

The remaining set of chains has the property that the second-highest element of every chain is in a particular set of permutations. We show that the Hall sum over the remaining chains is equivalent to summing the Möbius function over the set of permutations. The permutations in this set have the property that they can all be formed as the sum of  $\beta$  and either one, two or three copies of the permutation 1. For example, one of these permutations is  $1 \oplus \beta$ , and another is  $1 \ominus ((\beta \ominus 1) \oplus 1)$ . If we let  $\rho$  be one of the permutations in the set, then this means that we can use a well-known result to show that  $\mu[1, \rho] = \pm \mu[1, \beta]$ .

We use this to show that if  $\beta$  is a 2413-balloon, and  $\pi$  is the 2413-balloon of  $\beta$ , then  $\mu[1, \pi] = 2\mu[1, \beta]$ , and this in turn allows us to show that the growth of the principal Möbius function on the permutation poset is exponential.

We then consider 2413-balloons where the permutation being ballooned is not itself a 2413-balloon. Using a similar argument to that used for double 2413-balloons, we

derive an expression for  $\mu[1, \pi]$ , where  $\pi$  is the 2413-balloon of some permutation  $\beta$ , and  $\beta$  is not a 2413-balloon. For all but trivial cases we prove that  $\mu[1, \pi] = \mu[1, \beta]$ .

We discuss generalising the “balloon” operation. We provide two conjectures which, up to symmetry, cover all generalised 2413-balloons.

The chapter concludes with a brief discussion of a set of permutations where it is believed that the growth of the principal Möbius function is also exponential, but the “growth rate” is faster than that found for double 2413-balloons. We also discuss generalised balloons.

#### 1.4.4 The principal Möbius function of balloon permutations

Chapter 7 is based on an unpublished paper which is being prepared for submission in parallel with this thesis, and which is sole work by the author. In this paper we generalise the 2413-balloon permutations used in Chapter 6, and derive an expression for the value of the principal Möbius function of these permutations.

We start by defining a method of constructing a permutation from two smaller permutations  $\alpha$  and  $\beta$ . This construction method requires that the constructed permutation contains  $\beta$  as an interval copy, and that the remaining points are order-isomorphic to  $\alpha$ . We call such a permutation a “balloon” permutation, which we write as  $\alpha \odot_{i,j} \beta$ . We describe several sub-types of balloon permutation, including one that we call a “wedge” permutation.

We show that the chains in the poset interval  $[1, \alpha \odot_{i,j} \beta]$  can be partitioned into three sets. We further show that one of these subsets contributes zero to the value of the Möbius function. We then prove that the contribution of a second subset can be written as the sum of the principal Möbius function of a set of permutations, all of which contain  $\beta$  as an interval copy. This leads to an expression for  $\mu[1, \alpha \odot_{i,j} \beta]$ . This expression includes a “correction factor”, expressed as a sum over a particular set of (hard to handle) chains.

We then consider wedge permutations, and we show that the correction factor is always zero, thus leading to a simplified expression for the principal Möbius function of a wedge permutation.

We further show that the principal Möbius function of a wedge permutation is always a multiple of the principal Möbius function of  $\beta$ .

We discuss some of the problems that need to be overcome in order to extend our result to any interval where the upper bound is a balloon permutation.





# Chapter 2

## Common definitions

A *permutation* of length  $n$  is an ordering of the natural numbers  $1, \dots, n$ . For short (length less than 10) permutations, we write the permutation without delimiters, so 2413 represents a permutation of length 4, where the first element has value 2. For longer permutations, we use commas to delimit values, so, for example, we would write 1, 7, 4, 3, 10, 9, 2, 5, 8, 6. We may occasionally use commas as delimiters in short permutations, where this will aid the reader. We use  $\pi_i$  to refer to the  $i$ -th element of the permutation  $\pi$ , so, for example, if  $\pi = 2413$ , then  $\pi_1 = 2$ . We let  $\epsilon$  denote the unique permutation of length 0. We write the length of a permutation  $\pi$  as  $|\pi|$ .

If  $L$  is a list of distinct integers, then we can treat  $L$  as a permutation by replacing the  $i$ -th smallest entry with  $i$ . As an example, if  $L = 4, 9, 2, 6$ , then  $L$  represents the permutation 2413.

A permutation  $\pi$  can be represented graphically by plotting the points  $(i, \pi_i)$ , with  $1 \leq i \leq |\pi|$ , as shown in Figure 2.1. Throughout we treat a permutation and its plot interchangeably.

The set of all permutations of length  $n$  is written  $\mathcal{S}_n$ . A sequence of numbers  $a_1, a_2, \dots, a_n$  is *order-isomorphic* to a sequence  $b_1, b_2, \dots, b_n$  if for every  $i, j \in [1, n]$  we have  $a_i < a_j \Leftrightarrow b_i < b_j$ . A permutation  $\pi \in \mathcal{S}_n$  *contains* a permutation  $\sigma \in \mathcal{S}_k$  as a *pattern* if  $\pi$  has a subsequence of length  $k$  order-isomorphic to  $\sigma$ . We say that  $\pi$  *avoids*  $\sigma$  if  $\pi$  does not contain  $\sigma$ . There are other ways to define pattern containment,

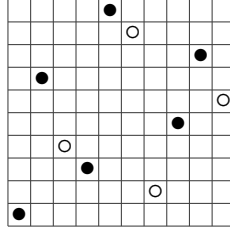


Figure 2.1: The permutation 1, 7, 4, 3, 10, 9, 2, 5, 8, 6, highlighting one of the copies of 2413 that it contains.

some of which are more specific, and others more general. We discuss some of these in Chapter 3, and in that chapter we refer to containment as defined above as *classic pattern containment*. Throughout the rest of this document we omit the qualifier “classic”.

As an example of containment, the permutation 2413 is contained in the permutation 1, 7, 4, 3, 10, 9, 2, 5, 8, 6, as shown in Figure 2.1.

If  $\sigma$  is contained in  $\pi$ , then there will be at least one set of points of  $\pi$ , with cardinality  $|\sigma|$ , such that the set of points is order-isomorphic to  $\sigma$ . We call such a set of points an *embedding* of  $\sigma$  into  $\pi$ . The points highlighted in Figure 2.1, (4,9,2,6), represent one possible embedding of 2413 in 1, 7, 4, 3, 10, 9, 2, 5, 8, 6. Typically, where embeddings are used, the arguments used require that only some of the embeddings are counted, and these are generally referred to as *normal embeddings*. It is notable that the precise definition of a normal embedding varies between papers, and indeed, in Brignall et al [15], several different definitions of normal embedding are used.

We note here that one problem with embeddings arises in cases such as  $\mu[1, 24153]$ . Here there are plainly only five ways to embed the permutation 1 into 24153, however  $\mu[1, 24153] = 6$ , and thus the embedding approach is not sufficient. One possible solution to this issue is to count the normal embeddings and then add a correction factor.

The set of all permutations, ordered by pattern containment, is a *poset* (partially ordered set).

If we have two permutations  $\sigma$  and  $\pi$  such that  $\sigma$  is not contained in  $\pi$ , and  $\pi$  is not

contained in  $\sigma$ , then we say that  $\sigma$  and  $\pi$  are *incomparable*.

A closed interval  $[\sigma, \pi]$  in a poset is the set defined as  $\{\tau : \sigma \leq \tau \leq \pi\}$ . A half-open interval  $[\sigma, \pi)$  is the set  $\{\tau : \sigma \leq \tau < \pi\}$ , and the open interval  $(\sigma, \pi)$  is the set  $\{\tau : \sigma < \tau < \pi\}$ . The *Möbius function*,  $\mu[\sigma, \pi]$ , is defined for an ordered pair of elements  $(\sigma, \pi)$  from any poset. If  $\sigma \not\leq \pi$ , then  $\mu[\sigma, \pi] = 0$ , and if  $\sigma = \pi$ , then  $\mu[\sigma, \pi] = 1$ . The remaining possibility is that  $\sigma < \pi$ , and in this case we have

$$\mu[\sigma, \pi] = - \sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda]. \quad (2.1)$$

If we have  $\sigma < \pi$ , then from the definition above we also have

$$\sum_{\lambda \in [\sigma, \pi]} \mu[\sigma, \lambda] = 0.$$

For posets that possess a unique smallest element  $\hat{0}$ , we define the *principal Möbius function*,  $\mu[\pi] = \mu[\hat{0}, \pi]$ . We will occasionally want to discuss the Möbius function of a poset  $P$  with a unique minimal element  $\hat{0}$ , and unique maximal element  $\hat{1}$ . Here, we define  $\mu[P] = \mu[\hat{0}, \hat{1}] = \sum_{\hat{0} \leq x < \hat{1}} \mu[\hat{0}, x]$ .

The *Hasse diagram* of a poset  $P$  is a directed graph, where two vertices  $u$  and  $w$  are connected from  $u$  to  $w$  by a directed arc if and only if  $w < u$ , and there is no  $v$  such that  $w < v < u$ . The Hasse diagram of the permutation poset  $[1, 13524]$  is shown in Figure 2.2. Note that we sometimes omit the arrowheads for clarity, as Hasse diagrams, by convention, are always drawn with the arc direction downwards.

A *chain* in a poset interval  $[\sigma, \pi]$  is, for our purposes, a subset of the elements in the interval  $[\sigma, \pi]$ , where the subset includes the elements  $\sigma$  and  $\pi$ , and every distinct pair of elements of the subset are comparable. This last clause means that the subset has a total order. If a chain  $c$  has  $k$  elements, then we say that the length of  $c$ , written  $|c|$ , is  $k - 1$ . One way to visualise a chain is to first choose a path in the Hasse diagram from the highest entry to the lowest, and then a chain is found by choosing a (possibly improper) subset of the elements on the path, ensuring that the first and last elements (the upper and lower bounds of the poset) are included in the subset.

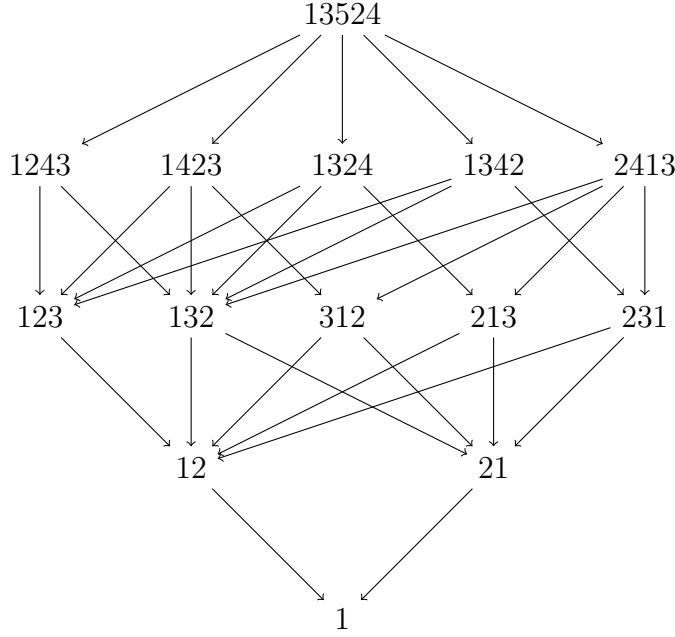


Figure 2.2: The Hasse diagram of the poset interval  $[1, 13524]$ .

Chains in a poset interval are related to the Möbius function by Hall's Theorem [50, Proposition 3.8.5], which says that

$$\mu[\sigma, \pi] = \sum_{c \in \mathcal{C}(\sigma, \pi)} (-1)^{|c|} = \sum_{i=1}^{|\pi|-1} (-1)^i K_i$$

where  $\mathcal{C}(\sigma, \pi)$  is the set of chains in the poset interval  $[\sigma, \pi]$ , and  $K_i$  is the number of chains of length  $i$ .

If  $\mathcal{C}$  is a subset of the chains in some poset interval  $[\sigma, \pi]$ , then the *Hall sum* of  $\mathcal{C}$  is  $\sum_{c \in \mathcal{C}} (-1)^{|c|}$ .

A *parity-reversing involution*,  $\Phi : \mathcal{C} \mapsto \mathcal{C}$ , is an involution such that for any  $c \in \mathcal{C}$ , the parities of  $c$  and  $\Phi(c)$  are different.

A simple corollary to Hall's Theorem is

**Corollary 1.** *If we can find a set of chains  $\mathcal{C}$  with a parity-reversing involution, then the Hall sum of  $\mathcal{C}$  is zero.*

---

*Proof.* Because there is a parity-reversing involution, the number of chains in  $\mathcal{C}$  with odd length is equal to the number of chains with even length, so  $\sum_{c \in \mathcal{C}} (-1)^{|c|} = 0$ .  $\square$

In Chapters 6 and 7 we will want to show that there is a parity-reversing involution on a set of chains  $\mathcal{C}$ . Our basic methodology, given a set of chains  $\mathcal{C}$ , and a chain  $c \in \mathcal{C}$ , will be to construct a chain  $c'$  by using a parity-reversing involution  $\Phi$ . Strictly speaking,  $\Phi$  is a function that maps a set of permutations (which is a chain) to a set of permutations (which may not be a chain). As examples, if  $\Phi(c)$  removes the largest or smallest element of  $c$ , or adds an element so that  $\Phi(c)$  does not have a total order, then  $\Phi(c)$  is not a chain. To show that  $\Phi$  is a parity-reversing involution we will need to show that  $\Phi(c)$  is a chain in  $\mathcal{C}$ , and that  $c$  and  $\Phi(c)$  have opposite parities. In our discussions, we will typically set  $c' = \Phi(c)$ , and then show that the set of permutations  $c'$  is a chain. We will then, without further comment, treat  $c'$  as a chain.

When discussing chains, in general we will only be interested in a small subset of the chain containing two or three elements. We say that a *segment* of some chain  $c$  is a non-empty subset of the elements in  $c$  with the property that any element not in the segment is either less than every element in the segment, or is greater than every element in the segment.

A *direct sum* of two permutations  $\alpha$  and  $\beta$  of lengths  $m$  and  $n$  respectively is the permutation  $\alpha_1, \dots, \alpha_m, \beta_1 + m, \dots, \beta_n + m$ . We write a sum as  $\alpha \oplus \beta$ . A *skew sum*,  $\alpha \ominus \beta$ , is the permutation  $\alpha_1 + n, \dots, \alpha_m + n, \beta_1, \dots, \beta_n$ . As examples,  $321 \oplus 213 = 321546$ , and  $321 \ominus 213 = 654213$ , and these are shown in Figure 2.3. A *sum-indecomposable* (resp. *skew-indecomposable*) permutation is a permutation that cannot be written as the direct sum (resp. skew sum) of two smaller permutations. If a permutation is not sum-indecomposable, then it is *sum-decomposable*, and if a permutation is not skew-indecomposable, then it is *skew-decomposable*,

Given a permutation  $\pi$ , the *finest* sum decomposition (resp. skew decomposition) of  $\pi$  is a decomposition into the maximum number of sum-indecomposable (resp. skew-indecomposable) permutations. As examples, using Figure 2.3, the finest sum decomposition of 321546 is  $321 \oplus 21 \oplus 1$ , and the finest skew-decomposition of 654213 is  $1 \ominus 1 \ominus 1 \ominus 213$ .

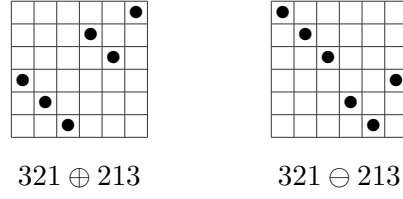


Figure 2.3: Examples of direct and skew sums.

Let  $\alpha$  be a permutation, and  $r$  a positive integer. Then  $\oplus^r \alpha$  is  $\alpha \oplus \alpha \oplus \dots \oplus \alpha \oplus \alpha$ , with  $r$  occurrences of  $\alpha$ . If  $S$  is a set of permutations, then  $\oplus^r S = \cup_{\lambda \in S} \{\oplus^r \lambda\}$ .

A *layered* permutation is a permutation that can be written as the direct sum of one or more decreasing permutations. Egge and Mansour, in [20], show that layered permutations can also be defined as permutations that avoid the permutations 231 and 312. The first example in Figure 2.3 is a layered permutation.

If a permutation  $\pi$  can be written as  $1 \oplus 1 \oplus \tau$ ,  $1 \ominus 1 \ominus \tau$ ,  $\tau \oplus 1 \oplus 1$ , or  $\tau \ominus 1 \ominus 1$ , where  $\tau$  is non-empty (so  $|\pi| \geq 3$ ), then we say that  $\pi$  has a *long corner*.

We will occasionally want to discuss situations where some permutation  $\pi$  is known to have a sum-decomposable (resp. skew-decomposable) decomposition, but we do not know exactly which permutations form the decomposition. In such cases we will write  $\pi = \alpha_1 \oplus \dots \oplus \alpha_n$  or  $\pi = \alpha_1 \ominus \dots \ominus \alpha_n$ . Similarly, if we want to discuss an arbitrary set of permutations then we will write  $\{\alpha_1, \dots, \alpha_n\}$ . It will always be clear from the context whether  $\alpha_i$  refers to the  $i$ -th element of the permutation  $\alpha$ , the  $i$ -th permutation in a sum, or the  $i$ -th permutation in a set of permutations.

An *interval* in a permutation  $\pi$  is a non-empty contiguous set of indexes  $i, i+1, \dots, j$  such that the set of values  $\{\pi_i, \pi_{i+1}, \dots, \pi_j\}$  is also contiguous. Every permutation  $\pi$  has intervals of length 1 and of length  $|\pi|$ , which we call *trivial intervals*. A *simple* permutation is a permutation that only has trivial intervals. As examples, 1324 is not simple, as, for example, the second and third points (32) form a non-trivial interval, whereas 2413 is simple.

We say that  $\pi$  has an *interval copy* of a permutation  $\alpha$  if it contains an interval of length  $|\alpha|$  whose elements form a subsequence order-isomorphic to  $\alpha$ .

We note here that the term “interval” is used in relation to both posets and permutations. This is standard terminology in the field, and when we use the term “interval” it will be clear from the context whether we are referring to a poset interval or an interval in a permutation.

An interval of length 2 is termed an *adjacency* in this thesis. An adjacency is clearly order-isomorphic to either 12, an *up-adjacency*, or to 21, a *down-adjacency*. If a permutation contains at least one up-adjacency and at least one down-adjacency, then we say that the permutation has *opposing adjacencies*. An interval of length 3 that is monotonic, that is, order-isomorphic to either 123 or 321, is a *triple-adjacency*. We note here that some sources use “adjacency” to refer to a non-trivial interval of any length that is monotonic.

A *descent* in a permutation  $\pi$  is a position  $i$  such that  $\pi_i > \pi_{i+1}$ . Similarly, an *ascent* in a permutation  $\pi$  is a position  $i$  such that  $\pi_i < \pi_{i+1}$ .

A *permutation class* is a set of permutations  $C$  with the property that if  $c \in C$ , and  $d < c$ , then  $d \in C$ . Every permutation class can be defined by the minimal set of permutations that are not contained in the class, and this minimal set is referred to as the *basis*. If a permutation class  $C$  has basis  $B$ , then we write  $C = \text{Av}(B)$ . Where we want to discuss a permutation class  $C$  that contains a specific set of (normally simple) permutations, we refer to  $C$  as a *hereditary class*. There is no difference between a permutation class and a hereditary class, the distinction is simply used to draw attention to the properties of the class that we are discussing.

Permutations can be represented by plotting points in a square grid, as described earlier. It is clear that any symmetry of the square, if applied to a permutation plot, will result in another permutation plot. If  $\alpha$  is a permutation, then a reflection in a vertical bisector of the plot is called a *reversal*, written  $\alpha^R$ , a reflection in a horizontal bisector of the plot is called a *complement*, written  $\alpha^C$ . The inverse of a permutation, written  $\alpha^{-1}$  is also a symmetry. These three operations are the generating set of the group of symmetries of permutations.

We can now see that for any permutations  $\sigma$  and  $\pi$ , and any symmetry  $S$ ,

$$\mu[\sigma, \pi] = \mu[\sigma^S, \pi^S].$$



We will occasionally want to discuss permutations where we want a unique representative from the symmetries. We say that such a representative is the *canonical* form of the permutation, and for our purposes we choose the symmetry which is smallest under the lexicographic order. As an example, 2413 and 3142 are symmetries of one another, and the canonical representation is 2413.

# Chapter 3

## Background and history

### 3.1 Permutations

The first, albeit implicit, reference to permutations in the literature appears to be due to Euler in [22], where he describes polynomials which essentially define what are now known as the Eulerian numbers  $A_{n,m}$ . In permutational terms,  $A_{n,m}$  counts the number of permutations of length  $n$  that have  $m$  descents. The next significant set of results comes some 150 years later, where MacMahon [30] has a result that, interpreted in permutational terms, shows that  $\text{Av}(123)$  is counted by the Catalan numbers. The Erdős-Szekeres theorem [21] can be interpreted as saying that a permutation of length  $(a-1)(b-1)+1$  must contain either an increasing sequence of length  $a$  or a decreasing sequence of length  $b$ .

The study of pattern avoidance in permutations can be said to have started with exercise 2.2.1(5) in Knuth [29], where readers are essentially asked to show that a permutation that can be stack-sorted must avoid 231. This work was further developed in the 1970s and 1980s in papers by Knuth [28], Rogers [35], Rotem [37], and Simion and Schmidt [42].

This initial development then turned into a veritable explosion of papers, most of which are too specific to relate to this general background. A good summary of the way in which the field has developed can be found in the book by Kitaev [27],

the book by Bona [10], the survey article by Steingrímsson [51], and the chapter by Vatter on permutation classes in [9].

## 3.2 The Möbius function

The Möbius function was first defined in the context of number theory by August Möbius in 1832 in [34]. In that paper, Möbius defines  $\mu(n) : \mathbb{N} \mapsto \mathbb{N}$  as 0 if  $n$  has a repeated prime factor, and as  $(-1)^k$  if  $n$  is the product of  $k$  distinct prime factors. If we say that a positive integer  $a$  is contained in a positive integer  $b$  if  $a$  divides  $b$ , then the integers under this relationship form a poset, and  $\mu(n) = \mu[1, n]$ .

The number-theoretic Möbius function has been extensively studied since its definition. The combinatorial Möbius function does not seem to have any significant presence in the literature until a seminal paper by Rota in 1964 [36], which made an explicit link between the principle of inclusion–exclusion and the combinatorial Möbius function.

While there are many papers that have results relating to the Möbius function on a variety of posets, we refer the reader to Cameron [19] or Stanley [50] for a general background to the area.

The classic definition of the Möbius function, as given in Equation 2.1, is, essentially, a recursive sum over the elements of the poset. This thesis, in general, restricts itself to this view. A simple consequence of the fundamental definition is Hall’s Theorem [50, Proposition 3.8.5] which defines the Möbius function as a sum over the chains in the poset. There are, however, other ways in which we can understand the Möbius function, and in order to provide a broad background, we briefly describe two of them here.

### 3.2.1 Simplicial complexes

Given a set of vertices  $V$ , a *simplicial complex*  $\Delta$  is a non-empty set of subsets of  $V$  such that if  $v \in V$ , then  $\{v\} \in \Delta$ ; and if  $G \in \Delta$ , and  $F \subset G$ , then  $F \in \Delta$ . If  $F \in \Delta$ , then we say that  $F$  has dimension  $|F| - 1$ . We then refer to  $F$  as a *face* of  $V$ . Note that the empty subset  $\emptyset$  is a face of  $V$ . If we have two elements of a poset  $\sigma$  and  $\pi$ , with  $\sigma < \pi$ , and  $(\sigma, \pi)$  is non-empty, then we can set  $V$  to be the set of chains in the open interval  $(\sigma, \pi)$ , and this gives a simplicial complex  $\Delta$ . Given a simplicial

complex  $\Delta$ , the reduced Euler characteristic of  $\Delta$ ,  $\chi(\Delta)$  is defined as

$$\chi(\Delta) = \sum_{k=-1}^{\dim \Delta} (-1)^k f_k(\Delta),$$

where  $f_k(\Delta)$  is the number of faces of dimension  $k$ . If we have a poset  $P$  with unique minimal and maximal elements  $\hat{0}$  and  $\hat{1}$  respectively, and set  $\Delta$  to be the chains in the open interval  $(\hat{0}, \hat{1})$ , then Hall's Theorem (see, for example, Stanley [50, Proposition 3.8.5] or Wachs [53, Proposition 1.2.6]) gives us that  $\mu[P] = \chi(\Delta)$ .

Since a simplicial complex is a topological entity, in addition to the possibility of using the Möbius function to determine the value of the reduced Euler characteristic, it is possible to pose questions about the topology of the poset. While this approach has been taken in some papers (discussed in Section 3.4 below), this thesis does not use this approach or provide any topological results. The interested reader is referred to the material in [53] for further details.

### 3.2.2 Incidence algebras and incidence matrices

A poset  $P$  is *locally finite* if, for every  $\sigma, \pi \in P$ , the interval  $[\sigma, \pi]$  has a finite number of elements. Following Rota [36], we define an *incidence algebra* by first taking a locally finite partially ordered set  $P$ , and considering the set of all real-valued functions  $f(x, y)$ , where  $x, y \in P$  and  $f(x, y) = 0$  if  $x \not\leq y$ . We then define the incidence algebra of  $P$  by convolution:

$$(f * g)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y).$$

This algebra has an identity element, normally written as  $\delta(x, y)$ , which is defined as

$$\delta(x, y) = \begin{cases} 1 & \text{If } x = y \\ 0 & \text{Otherwise.} \end{cases}$$

The *zeta function* is defined as

$$\zeta(x, y) = \begin{cases} 1 & \text{If } x \leq y \\ 0 & \text{Otherwise.} \end{cases}$$

With these definitions, it can be shown that the Möbius function is the convolutional inverse of  $\zeta$ , so

$$\zeta * \mu(x, y) = \mu * \zeta(x, y) = \delta(x, y).$$

Let  $Z$  be a square matrix, with rows and columns indexed by the elements of a poset  $P$ , and with  $Z_{x,y} = \zeta(x, y)$ . We call this the *zeta matrix* of  $P$ . It is now possible to show that if  $x$  and  $y$  are elements of the poset, then

$$\mu[x, y] = (Z^{-1})_{x,y}.$$

We remark here that the result above implies that we can determine the value of the Möbius function for every interval in a poset by (simply) calculating the inverse of the zeta matrix. We have some computational evidence that using the “matrix inverse” method to determine the value of the Möbius function for a significant number of intervals in a poset is computationally more efficient than using the fundamental definition given in Equation 2.1. On the other hand, if we want to determine the value of the Möbius function for a single interval, then the fundamental definition seems to be significantly faster than the matrix inverse method. Of course, our ideal is to find methods that can determine the value of the Möbius function faster than either the matrix inverse method, or using the fundamental definition.

### 3.3 The Möbius function for general posets

Before we move on to consider the Möbius function of the permutation poset under classic pattern containment, we divert slightly to review some results relating to the Möbius function on other posets. We start by remarking that, for a general poset, using the recursive definition of the Möbius function is computationally hard. Our purpose in this section is to establish that determining the Möbius function need not be computationally hard in some cases. We refer the reader to [27] for a good overview of most of the containment types discussed in this section.

There are some well-known cases where an explicit formula exists for the Möbius function. For example, the Möbius function on a Boolean algebra is given by  $\mu[R, S] = (-1)^{\#(S-R)}$  (see, for instance, Example 3.8.3 in [50]).

A slightly more complex example is given by the poset of subspaces of a vector space  $V \subseteq GF(q)^n$ . If we have  $U \subseteq W \subseteq V$ , then

$$\mu[U, W] = (-1)^k q^{\binom{k}{2}}, \text{ where } k = \dim(W) - \dim(U).$$

This result is attributed to Hall in Rota [36]. The poset of subspaces of a vector space is an example of a lattice, and the proof given in Rota utilises this fact. The Möbius function for general lattices is also well-known (see, for instance, Section 3.9 in [50]).

We now turn to posets that are defined by free monoids over alphabets, or by permutations using a containment other than classic pattern containment.

Björner completely determined the Möbius function of subword order in [7], and then completely determined the Möbius function for factor order in [8].

Sagan and Vatter, in [39], considered ordered partitions (compositions) of an integer, with a partial order given by subwords, and completely determined the Möbius function on this poset. This paper also has the first result for the permutation poset under classic pattern containment, which we discuss in Section 3.4.

Bernini, Ferrari and Steingrímsson, in [6], considered permutations using consecutive

pattern containment. For most intervals  $[\sigma, \pi]$  they have a set of explicit formulae for  $\mu[\sigma, \pi]$ , based mainly on how many times  $\sigma$  occurs in  $\pi$ . For intervals not covered by their formulae, they provide a polynomial algorithm to calculate the Möbius function. We consider that the Möbius function of permutations under consecutive pattern containment is, therefore, completely known. Sagan and Willenbring, in [40], reproduced this result using a technique known as discrete Morse theory.

Bernini and Ferrari, in [5], introduced the quasi-consecutive pattern poset of permutations, where  $\sigma$  is contained in  $\pi$  if  $\pi$  contains an occurrence of  $\sigma$  where all entries are adjacent, except possibly the first and second. They completely determine the Möbius function for any interval  $[\sigma, \pi]$  where  $\sigma$  occurs exactly once in  $\pi$ .

A recent preprint by Bernini, Cervetti, Ferrari and Steingrímsson [4] considers the poset of Dyck paths, where we say that a path  $P$  contains a path  $Q$  if the steps in  $Q$  are a subsequence of the steps in  $P$ . The preprint includes expressions for the Möbius function of some specific intervals in this poset.

The posets discussed so far are, in some way, simpler than the permutation poset under classic pattern containment, and we have seen that the Möbius function has either been completely determined, or, as in the last two examples, has been determined for a particular subset of intervals in the poset. We now consider posets that, in some sense, generalize classic pattern containment.

Mesh patterns are a generalization of classic pattern containment on permutations, and the poset of mesh patterns contains the poset of permutations as an induced subposet. We refer the reader to [11] for a formal definition of mesh patterns. In [49], Smith and Ulfarsson present some initial results on the Möbius function of the mesh pattern poset, and show that as  $n \rightarrow \infty$ , the proportion of mesh patterns  $p$  of length  $n$  with  $\mu[1^\emptyset, p] = 0$  approaches 1, where  $1^\emptyset$  is the unshaded singleton mesh pattern. The mesh pattern  $1^\emptyset$  corresponds to the permutation 1 in the induced poset of permutations. In the permutation pattern poset, we know [2] that the number of simple permutations of length  $n$  is, asymptotically,  $\frac{n!}{e^2}$ , and it is generally believed that for most simple permutations the value of the principal Möbius function is non-zero, thus in the permutation pattern poset we do not expect the proportion of permutations  $\pi$  of length  $n$  with  $\mu[\pi] = 0$  to approach 1.



Smith generalised pattern containment in [48], and found some explicit formulae for the Möbius function. These formulae have the general form

$$\mu[\sigma, \pi] = (-1)^{|\pi| - |\sigma|} E(\sigma, \pi) + \sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda] \mu[\hat{P}(\lambda, \pi)],$$

where  $E(\sigma, \pi)$  counts specific types of embeddings of  $\sigma$  into  $\pi$ , and  $\hat{P}(\lambda, \pi)$  is a poset derived from the interval  $[\lambda, \pi]$ .

Although this last example does have a complete characterisation of the Möbius function on all intervals of the poset, from a computational perspective the result is only useful if, outside the result given, we can show that the second term is zero. This result is a generalisation of some of the results given by Smith in [46], which we discuss in the following section.

### 3.4 The Möbius function of the permutation poset under pattern containment

The study of the Möbius function in the (classic) permutation poset was introduced by Wilf [54], who wrote

We can partially order the set of all permutations of all numbers of letters by declaring that  $\sigma \leq \tau$  if  $\sigma$  is contained as a pattern in  $\tau$ . It would be interesting to study this as a poset. For example, what can be said about its Möbius function?

The first result in this area was by Sagan and Vatter [39]. Their paper primarily concerns itself with the Möbius function of a composition poset. An integer composition can be thought of as an ordered list of positive integers, and a layered permutation can be completely specified by such a list, so there is a bijection between integer compositions and layered permutations. From this it follows that there is a bijection between the poset of compositions of integers and the poset of layered permutations. In the final section of their paper, they use this bijection to essentially give an expression for the Möbius function on intervals in the poset of layered permutations under classic pattern containment.

Steingrímsson and Tenner [52] found a large class of pairs of permutations  $(\sigma, \pi)$  where  $\mu[\sigma, \pi] = 0$ . They show that the (poset) interval  $[\sigma, \pi]$  has  $\mu[\sigma, \pi] = 0$  if  $\pi$  contains a non-trivial interval where none of the elements of the (permutation) interval are part of an embedding of  $\sigma$  into  $\pi$ . They also show that if there is exactly one embedding of  $\sigma$  into  $\pi$ , and the complement of the embedding satisfies certain conditions, then  $\mu[\sigma, \pi] \in \{0, \pm 1\}$ .

In a seminal paper, Burstein, Jelínek, Jelínková and Steingrímsson [18] found a recursion for the Möbius function for sum/skew decomposable permutations in terms of the sum/skew indecomposable permutations in the lower and upper bounds. They also found a method to determine the Möbius function for separable permutations

by counting embeddings. The recursions for decomposable permutations are used extensively in this thesis.

McNamara and Steingrímsson [33] investigated the topology of intervals in the permutation poset, and in doing so found a single recurrence equivalent to the recursions in [18].

We now compare the recursions from Burstein et al [18] with those from McNamara and Steingrímsson [33]. The recursions from Burstein, Jelínek, Jelínková and Steingrímsson [18] can be written as follows.

**Proposition 2** (McNamara and Steingrímsson [33, Proposition 8.3], following Burstein, Jelínek, Jelínková and Steingrímsson [18, Proposition 1]). *Let  $\sigma$  and  $\pi$  be non-empty permutations with finest decompositions  $\sigma = \sigma_1 \oplus \dots \oplus \sigma_s$  and  $\pi = \pi_1 \oplus \dots \oplus \pi_t$ , where  $t \geq 2$ . Suppose that  $\pi_1 = 1$ . Let  $k \geq 1$  be the largest integer such that all the components  $\pi_1, \dots, \pi_k$  are equal to 1, and let  $\ell \geq 0$  be the largest integer such that all the components  $\sigma_1, \dots, \sigma_\ell$  are equal to 1. Then*

$$\mu[\sigma, \pi] = \begin{cases} 0 & \text{if } k - 1 > \ell, \\ -\mu[\sigma_{>k-1}, \pi_{>k}] & \text{if } k - 1 = \ell, \\ \mu[\sigma_{>k}, \pi_{>k}] - \mu[\sigma_{>k-1}, \pi_{>k}] & \text{if } k - 1 < \ell. \end{cases}$$

The remaining case is  $\pi_1 > 1$ , and is covered by the next proposition.

**Proposition 3** (McNamara and Steingrímsson [33, Proposition 8.4], following Burstein, Jelínek, Jelínková and Steingrímsson [18, Proposition 2]). *Let  $\sigma$  and  $\pi$  be non-empty permutations with finest decompositions  $\sigma = \sigma_1 \oplus \dots \oplus \sigma_s$  and  $\pi = \pi_1 \oplus \dots \oplus \pi_t$ , where  $t \geq 2$ . Suppose that  $\pi_1 > 1$ . Let  $k \geq 1$  be the largest integer such that all the  $\pi_1, \dots, \pi_k$  are equal to  $\pi_1$ . Then*

$$\mu[\sigma, \pi] = \sum_{i=1}^s \sum_{j=1}^k \mu[\sigma_{\leq i}, \pi_1] \mu[\sigma_{>i}, \pi_{>j}].$$

The recursion found by McNamara and Steingrímsson can be written as follows.

**Proposition 4** (McNamara and Steingrímsson [33, Proposition 8.1]).

*Consider permutations  $\sigma$  and  $\pi$  and let  $\pi = \pi_1 \oplus \dots \oplus \pi_t$  be the finest decomposition of  $\pi$ . Then*

$$\mu[\sigma, \pi] = \sum_{\sigma = \varsigma_1 \oplus \dots \oplus \varsigma_t} \prod_{1 \leq m \leq t} \begin{cases} \mu[\varsigma_m, \pi_m] + 1 & \text{If } \varsigma_m = \epsilon \text{ and } \pi_{m-1} = \pi_m, \\ \mu[\varsigma_m, \pi_m] & \text{otherwise,} \end{cases}$$

*where the sum is over all direct sums  $\sigma = \varsigma_1 \oplus \dots \oplus \varsigma_t$ , such that  $\epsilon \leq \varsigma_m \leq \pi_m$  for all  $1 \leq m \leq t$ .*

We remark here that the recursion from McNamara and Steingrímsson is, in some sense, a nicer recursion than that found by Burstein, Jelínek, Jelínková and Steingrímsson. Despite this, in this thesis we use the Burstein et al recursions as these are easier to work with in the context of our results.

Smith [44] found an explicit formula for the Möbius function on the interval  $[1, \pi]$  for all permutations  $\pi$  with a single descent. Smith [45] has explicit expressions for the Möbius function  $\mu[\sigma, \pi]$  when  $\sigma$  and  $\pi$  have the same number of descents. In [46], Smith found an expression that determines the Möbius function for all intervals in the poset. The main result is

$$\mu[\sigma, \pi] = (-1)^{|\pi| - |\sigma|} \text{NE}(\sigma, \pi) + \sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda] \sum_{S \in EZ^{\lambda, \pi}} (-1)^{|S|} \quad (3.1)$$

where  $\text{NE}(\sigma, \pi)$  is the number of normal embeddings of  $\sigma$  into  $\pi$ , and  $EZ^{\lambda, \pi}$  is a set of sets of embeddings of  $\lambda$  into  $\pi$  that satisfy a particular condition.

One view of this result is that it tells us that the value of the Möbius function on an interval  $[\sigma, \pi]$  is given by the number of normal embeddings of  $\sigma$  into  $\pi$ , plus a correction factor. Smith notes [46, Remark 22] that 95% of intervals with  $|\pi| \leq 8$  have  $\mu[\sigma, \pi] = (-1)^{|\pi| - |\sigma|} \text{NE}(\sigma, \pi)$ , so in these cases the correction factor is zero.

Smith remarks [46, Remark 23] that where we can show that

$$\sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda] \sum_{S \in EZ^{\lambda, \pi}} (-1)^{|S|} = 0,$$

the normal embedding approach can determine the value of the Möbius function in polynomial time, whereas using the recursive formula of Equation 2.1 is exponential complexity.

One approach to showing that  $\sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda] \sum_{S \in EZ^{\lambda, \pi}} (-1)^{|S|} = 0$  would be to find permutations  $\sigma$  and  $\pi$  such that for any  $\lambda \in [\sigma, \pi)$ ,  $EZ^{\lambda, \pi}$  is empty, as this would force the second term to be zero. Some small-scale experiments by the author suggest that it is significantly more likely that some of the sets  $EZ^{\lambda, \pi}$  are non-empty, and therefore when  $\sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda] \sum_{S \in EZ^{\lambda, \pi}} (-1)^{|S|} = 0$  it is likely to be because, taken across every  $\lambda \in [\sigma, \pi)$ , the number of sets in  $EZ^{\lambda, \pi}$  with even order is the same as the number of sets with odd order.

Brignall and Marchant [16] showed that if the lower bound of an interval is indecomposable, then the Möbius function depends only on the indecomposable permutations contained in the upper bound. They then used this result to find a fast polynomial algorithm for computing  $\mu[\pi]$  where  $\pi$  is an increasing oscillation. This paper forms the basis of Chapter 4 of this thesis.

Brignall, Jelínek, Kynčl and Marchant [15] prove that if a permutation  $\pi$  contains opposing adjacencies, then  $\mu[\pi] = 0$ . They then use this to show that the proportion of permutations of length  $n$  with principal Möbius function equal to zero is asymptotically bounded below by  $(1 - 1/e)^2 \geq 0.3995$ . This paper forms the basis of Chapter 5 of this thesis.

Jelínek, Kantor, Kynčl and Tancer [25] show how to construct a sequence of permutations  $\pi_n$  with length  $2n + 2$ , and they show that for  $n \geq 2$ ,

$$\mu[\pi_n] = -\binom{n+2}{7} - \binom{n+1}{7} + 2\binom{n+2}{5} - \binom{n+2}{3} - \binom{n}{2} - 2n,$$

and thus demonstrate that the absolute value of the Möbius function grows according to the seventh power of the length. In their paper they also show that if  $f : [\sigma, \pi] \rightarrow \mathbb{R}$

is any function satisfying  $f(\pi) = 1$ , then

$$\mu[\sigma, \pi] = f(\sigma) - \sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda] \sum_{\tau \in [\lambda, \pi]} f(\tau)$$

and as a corollary, they then show that

$$\mu[\sigma, \pi] = (-1)^{|\pi| - |\sigma|} E(\sigma, \pi) - \sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda] \sum_{\tau \in [\lambda, \pi]} (-1)^{|\pi| - |\tau|} E(\tau, \pi),$$

where  $E(\alpha, \beta)$  is the number of embeddings of  $\alpha$  into  $\beta$ . We note that the formula in the corollary has a similar structure to Equation 3.1 described above, although in general it suffers from the same restrictions as Smith's equation.

Finally, Marchant [31] showed how to construct a sequence of permutations  $\pi_1, \pi_2, \pi_3, \dots$  with lengths  $n, n+4, n+8, \dots$  such that  $\mu[1, \pi_{i+1}] = 2\mu[1, \pi_i]$ , and this gives us that the growth of the principal Möbius function on the permutation poset is exponential. This paper forms the basis of Chapter 6 of this thesis.

### 3.5 Motivation

It seems reasonably clear that the Möbius function of the permutation poset under classic pattern containment is a non-trivial problem. This contrasts with some of the posets described in 3.3, where the Möbius function is completely determined.

As we have described, the study of the permutation poset under classic pattern containment was initiated by Wilf in 2002 in [54]. Anecdotally, Wilf is believed to have later said that the Möbius function on the permutations pattern poset was “*A mess. Don’t touch it*”.

We state here that we think that Wilf’s reported view is somewhat pessimistic. While we think that it is unlikely that there is a polynomial-time procedure for computing the value of the Möbius on an arbitrary interval of the permutation poset, we believe, and we hope to show in this thesis, that there is considerable scope for further research in this area.

The permutation pattern poset is the subject of considerable research activity outside of the Möbius function, and we claim that the permutation pattern poset is the underlying object for many studies related to patterns in permutations. This then means that research into the Möbius function on the permutation pattern poset may lead to a better understanding of this poset, and hence to results in other areas related to permutation patterns.

We can summarise our motivation for research in this area by saying that the Möbius function of the permutation poset under classic pattern containment is not well-understood, and indeed up until recently the proportion of permutations where we had a (computationally) simple way to determine the value of the principal Möbius function was, asymptotically, zero.

The paper which forms the basis of Chapter 5 shows that, asymptotically, the proportion of permutations where the principal Möbius function is zero is at least 0.3995. The corollary to this result, however, is that we do not yet have an effective means to compute the principal Möbius function for 60% of all permutations.

---

Further, research into the Möbius function on the permutation pattern poset may lead to a better understanding of the intrinsic properties of the poset, which in turn may lead to results in related areas.





## Chapter 4

# The Möbius function of permutations with an indecomposable lower bound

### 4.1 Preamble

This chapter is based on a published paper [\[16\]](#), which is joint work with Robert Brignall.

In this chapter, we show that the Möbius function of an interval in a permutation poset where the lower bound is sum (resp. skew) indecomposable depends solely on the sum (resp. skew) indecomposable permutations contained in the upper bound, and that this can simplify the calculation of the Möbius sum. For increasing oscillations, we give a recursion for the Möbius sum which only involves evaluating simple inequalities.

## 4.2 Introduction

Recall that the Möbius function on a poset interval  $[\sigma, \pi]$  is defined by

$$\mu[\sigma, \pi] = \begin{cases} 1 & \text{if } \sigma = \pi, \\ -\sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda] & \text{otherwise.} \end{cases} \quad (4.1)$$

Our motivation for this chapter is to find a *contributing set*  $\mathfrak{C}_{\sigma, \pi}$  that is significantly smaller than the poset interval  $[\sigma, \pi)$ , and a  $\{0, \pm 1\}$  weighting function  $W(\sigma, \alpha, \pi)$  such that

$$\mu[\sigma, \pi] = - \sum_{\alpha \in \mathfrak{C}_{\sigma, \pi}} \mu[\sigma, \alpha] W(\sigma, \alpha, \pi). \quad (4.2)$$

Plainly, in Equation 4.2, we could set  $\mathfrak{C}_{\sigma, \pi} = [\sigma, \pi)$ , and  $W(\sigma, \alpha, \pi) = 1$ , which is equivalent to Equation 4.1.

One approach here would be to take a permutation  $\beta$  such that  $\sigma < \beta < \pi$ . We could then set  $\mathfrak{C}_{\sigma, \pi} = \{\lambda : \lambda \in [\sigma, \pi) \text{ and } \lambda \notin [\sigma, \beta]\}$ , and  $W(\sigma, \alpha, \pi) = 1$ , since, from Equation 4.2,  $\sum_{\lambda \in [\sigma, \beta]} \mu[\sigma, \lambda] = 0$ . This approach was used in Smith [44], who determined the Möbius function on the interval  $[1, \pi]$  for all permutations  $\pi$  with a single descent. Smith's paper is unusual, in that it provides an explicit formula for the value of the Möbius function.

Our approach is different. We identify individual elements (say  $\lambda$ ), of the poset that have  $\mu[\sigma, \lambda] = 0$ . We also show that there are pairs of elements,  $\lambda$  and  $\lambda'$ , where  $\mu[\sigma, \lambda] = -\mu[\sigma, \lambda']$ , and so we can exclude these pairs of elements. Finally, we show that there are quartets of permutations  $\lambda_1, \dots, \lambda_4$  where  $\sum_{i=1}^4 \mu[\sigma, \lambda_i] = 0$ ; and that we can systematically identify these quartets. By excluding these permutations from  $\mathfrak{C}_{\sigma, \pi}$  we can significantly reduce the number of elements in  $\mathfrak{C}_{\sigma, \pi}$  compared to the number of elements in the interval  $[\sigma, \pi)$ . This approach results in the ability to compute  $\mu[\sigma, \pi]$ , where  $\sigma$  is indecomposable, much faster than evaluating Equation 4.1. For increasing oscillations, we will show that the elements of  $\mathfrak{C}_{\sigma, \pi}$  can be determined using simple inequalities, and that as a consequence  $\mu[\sigma, \pi]$  can be determined using

inequalities. With this approach, we have computed  $\mu[1, \pi]$ , where  $\pi$  is an increasing oscillation, up to  $|\pi| = 2,000,000$ .

Our main tool in the first part of this chapter comes from the results of Burstein, Jelínek, Jelínková and Steingrímsson [18]. They found a recursion for the Möbius function for sum/skew decomposable permutations in terms of the sum/skew indecomposable permutations in the lower and upper bounds. They also found a method to determine the Möbius function for separable permutations by counting embeddings. We use the recursions for decomposable permutations to underpin the first part of this chapter.

In this chapter we show that the Möbius function on intervals with a sum indecomposable lower bound depends only on the sum indecomposable permutations contained in the upper bound. We provide a weighting function that determines which sum indecomposable permutations contribute to the Möbius sum. We then consider increasing oscillations. For these permutations, we show how we can find all of the permutations that contribute to the Möbius sum by applying simple numeric inequalities, which leads to a fast polynomial algorithm for determining the Möbius function.

We start with some essential definitions and notation relevant to this chapter in Section 4.3, then in Section 4.4 we provide a number of preliminary lemmas. We conclude this section with a theorem that gives  $\mu[\sigma, \pi]$ , where  $\sigma$  is a sum indecomposable permutation, for all  $\pi$ . In Section 4.5 we consider  $\mu[\sigma, \pi]$  where  $\sigma$  is a sum indecomposable permutation, and  $\pi$  is an increasing oscillation. We finish with some concluding remarks in Section 4.6.

### 4.3 Definitions and notation

When discussing the Möbius function,  $\mu[\sigma, \pi] = -\sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda]$ , we will frequently be examining the value of  $\mu[\sigma, \lambda]$  for a specific permutation  $\lambda$ . We say that this is the *contribution* that  $\lambda$  makes to the sum. If we have a set of permutations  $S \subseteq [\sigma, \pi)$  such that  $\sum_{\lambda \in S} \mu[\sigma, \lambda] = 0$ , then we say that the set  $S$  makes *no net contribution* to the sum.

The *interleave* of two permutations  $\alpha$  and  $\beta$  is formed by taking the sum  $\alpha \oplus \beta$ , and then exchanging the value of the largest point from  $\alpha$  with the value of the smallest point from  $\beta$ . We can also view this as increasing the largest point from  $\alpha$  by 1, and simultaneously decreasing the smallest point from  $\beta$  by 1. We write an interleave as  $\alpha \odot \beta$ . For example,  $321 \odot 213 = 421536$ , see Figure 4.1.

For completeness, we also define a *skew interleave*,  $\alpha \oslash \beta$ , which is formed by taking the skew sum  $\alpha \ominus \beta$ , and then exchanging the smallest point from  $\alpha$  with the largest point from  $\beta$ . As an example,  $321 \oslash 213 = 653214$ , as shown in Figure 4.1.

The interleave operations,  $\odot$  and  $\oslash$ , are not associative, as  $1 \odot 1 \odot 1$  could represent 231 or 312. To avoid this ambiguity, we require that the permutation 1 can either be interleaved to the left or to the right, but not both. It is easy to see that this restriction establishes associativity. We note here that, with this restriction, an expression involving  $\oplus$  and  $\odot$  represents a unique permutation regardless of the order in which the operations are applied.

Let  $\alpha$  be a permutation with length greater than 1. We will frequently want to refer to permutations that have the form  $\alpha \odot \alpha \odot \dots \odot \alpha \odot \alpha$ . If there are  $n$  copies of  $\alpha$  being interleaved, then we will write this as  $\odot^n \alpha$ , so, for example, we have  $\odot^3(21) = 21 \odot 21 \odot 21 = 315264$ .

For the remainder of this chapter, by symmetry it suffices to discuss permutations in relation to sums and interleaves only. For the same reason, references to (in)decomposable permutations may omit the “sum” qualifier.

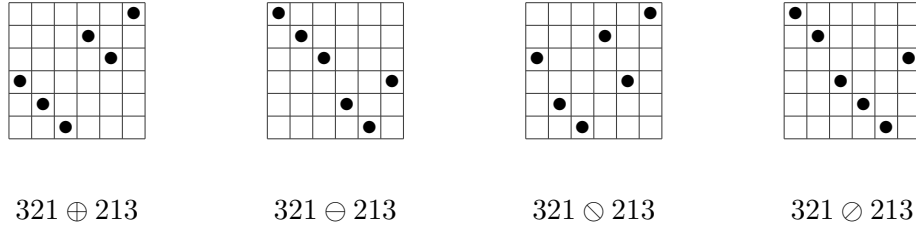


Figure 4.1: Examples of direct and skew sums and interleaves.

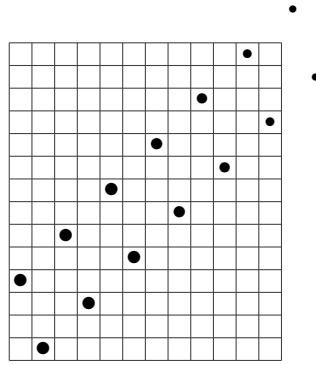


Figure 4.2: A depiction of the start of the increasing oscillating sequence.

The *increasing oscillating sequence* is the sequence

$$4, 1, 6, 3, 8, 5, 10, 7, \dots, 2k + 2, 2k - 1, \dots$$

The start of the sequence is depicted in Figure 4.2. An *increasing oscillation* is a simple permutation contained in the increasing oscillating sequence. For lengths greater than three, there are exactly two increasing oscillations of each length. Let  $W_n$  be the increasing oscillation with  $n$  elements which starts with a descent, and let  $M_n$  be the increasing oscillation with  $n$  elements which starts with an ascent. Then

$$\begin{aligned} W_{2n} &= \odot^n 21, & M_{2n} &= 1 \odot (\odot^{n-1} 21) \odot 1, \\ W_{2n-1} &= (\odot^{n-1} 21) \odot 1, & \text{and} & & M_{2n-1} &= 1 \odot (\odot^{n-1} 21). \end{aligned}$$

Note that  $W_n = M_n^{-1}$ .

There are instances where, for some permutation  $\alpha$ , we are interested in the set of permutations  $\{\alpha, 1 \oplus \alpha, \alpha \oplus 1, 1 \oplus \alpha \oplus 1\}$ . Given a permutation  $\alpha$ , we refer to this set as  $\mathcal{F}_\oplus(\alpha)$ , and we say that this set is the *family* of  $\alpha$ . If  $S$  is a set of permutations, then  $\mathcal{F}_\oplus(S) = \cup_{\alpha \in S} \{\mathcal{F}_\oplus(\alpha)\}$ .

There are also some instances where we are interested in the set of permutations  $\mathcal{F}_\odot(\alpha) = \{\alpha, 1 \odot \alpha, \alpha \odot 1, 1 \odot \alpha \odot 1\}$ . Note that every increasing oscillation is an element of  $\mathcal{F}_\odot(\odot^k 21)$  for some  $k \geq 1$ .

## 4.4 Preliminary lemmas and main theorem

In this section our aim is to show that if  $\sigma$  is indecomposable, then for any  $\pi \geq \sigma$  there is a  $\{0, \pm 1\}$  weighting function  $W(\sigma, \alpha, \pi)$  and a set of permutations  $\mathfrak{C}_{\sigma, \pi}$ , such that

$$\mu[\sigma, \pi] = - \sum_{\alpha \in \mathfrak{C}_{\sigma, \pi}} \mu[\sigma, \alpha] W(\sigma, \alpha, \pi).$$

If  $\pi$  is the identity permutation  $1, 2, \dots, n$  or its reverse, then  $\mu[\sigma, \pi]$  is trivial for any  $\sigma$ , and we exclude the identity and its reverse from being the upper bound of any interval under consideration.

As noted earlier, our approach is to show that there are permutations, pairs of permutations, and quartets of permutations in  $[\sigma, \pi)$  that make no net contribution to the sum.

We use Proposition 1 and 2, and Corollary 3 from Burstein, Jelínek, Jelínková and Steingrímsson [18]. Note that we have already introduced these propositions on page 26, but we repeat them here for ease of use. We start with some required notation. If  $\pi$  is a non-empty permutation with decomposition  $\pi_1 \oplus \dots \oplus \pi_n$ , then for any integer  $i$  with  $0 \leq i \leq n$ ,  $\pi_{\leq i}$  is the permutation  $\pi_1 \oplus \dots \oplus \pi_i$ , and  $\pi_{> i}$  is the permutation  $\pi_{i+1} \oplus \dots \oplus \pi_n$ . An empty sum of permutations is defined as  $\varepsilon$ , and in particular  $\pi_{\leq 0} = \pi_{> n} = \varepsilon$ . We can see that  $\mu[\varepsilon, \varepsilon] = 1$ ,  $\mu[\varepsilon, 1] = -1$  and  $\mu[\varepsilon, \tau] = 0$  for any  $\tau > 1$ . We now recall the results from Burstein, Jelínek, Jelínková and Steingrímsson:

**Proposition 5** (Burstein, Jelínek, Jelínková and Steingrímsson [18, Proposition 1]). *Let  $\sigma$  and  $\pi$  be non-empty permutations with decompositions  $\sigma = \sigma_1 \oplus \dots \oplus \sigma_m$  and  $\pi = \pi_1 \oplus \dots \oplus \pi_n$ , with  $n \geq 2$ . Assume that  $\pi_1 = 1$ , and let  $k$  be the largest integer such that  $\pi_1, \pi_2, \dots, \pi_k$  are all equal to 1. Let  $l \geq 0$  be the largest integer such that  $\sigma_1, \sigma_2, \dots, \sigma_l$  are all equal to 1. Then*

$$\mu[\sigma, \pi] = \begin{cases} 0 & \text{if } k - 1 > l, \\ -\mu[\sigma_{> k-1}, \pi_{> k}] & \text{if } k - 1 = l, \\ \mu[\sigma_{> k}, \pi_{> k}] - \mu[\sigma_{> k-1}, \pi_{> k}] & \text{if } k - 1 < l. \end{cases}$$



**Proposition 6** ([18, Proposition 2]). *Let  $\sigma$  and  $\pi$  be non-empty permutations with decompositions  $\sigma = \sigma_1 \oplus \dots \oplus \sigma_m$  and  $\pi = \pi_1 \oplus \dots \oplus \pi_n$ , with  $n \geq 2$ . Assume that  $\pi_1 \neq 1$ , and let  $k$  be the largest integer such that  $\pi_1, \pi_2, \dots, \pi_k$  are all equal to  $\pi_1$ . Then*

$$\mu[\sigma, \pi] = \sum_{i=1}^m \sum_{j=1}^k \mu[\sigma_{\leq i}, \pi_1] \mu[\sigma_{> i}, \pi_{> j}].$$

**Corollary 7** ([18, Corollary 3]). *Let  $\sigma$  and  $\pi$  be as in Proposition 6. Suppose that  $\sigma$  is sum indecomposable, so  $m = 1$ . Then*

$$\mu[\sigma, \pi] = \begin{cases} \mu[\sigma, \pi_1] & \text{if } \pi = \oplus^k \pi_1, \\ -\mu[\sigma, \pi_1] & \text{if } \pi = (\oplus^k \pi_1) \oplus 1, \\ 0 & \text{otherwise,} \end{cases}$$

A simple consequence of Propositions 5 and 6 is the identification of some intervals of permutations where the value of the Möbius function is zero.

**Lemma 8.** *Let  $\pi \in \{1 \oplus 1 \oplus \tau, \tau \oplus 1 \oplus 1, \mathcal{F}_{\oplus}((\oplus^r \alpha) \oplus \tau')\}$ , where  $\tau$  is any permutation,  $r$  is maximal,  $\alpha$  is sum indecomposable, and  $\tau'$  is any permutation greater than 1. Let  $\sigma$  be a sum indecomposable permutation. Then  $\mu[\sigma, \pi] = 0$ .*

*Proof.* Consider  $\pi = 1 \oplus 1 \oplus \tau$ . We use Proposition 5. If  $\tau_1 = 1$ , then  $k \geq 3$ , and  $l \leq 1$ , and the result follows immediately. Now assume that  $\tau_1 \neq 1$ . Then  $k = 2$ . If  $\sigma > 1$ , then again the result follows immediately. If  $\sigma = 1$ , then we have  $\mu[\sigma, \pi] = -\mu[\sigma_{> k-1}, \pi_{> k}] = -\mu[\varepsilon, \tau] = 0$ . The case for  $\pi = \tau \oplus 1 \oplus 1$  follows by symmetry.

Now consider  $\pi = \mathcal{F}_{\oplus}((\oplus^r \alpha) \oplus \tau')$ . If  $\pi = (\oplus^r \alpha) \oplus \tau$ , or  $\pi = (\oplus^r \alpha) \oplus \tau \oplus 1$ , then we use Proposition 6. In that context we have  $m = 1$  and  $k = r$ , and so  $\mu[\sigma, \pi] = \sum_{j=1}^r \mu[\sigma, \pi_1] \mu[\varepsilon, \pi_{> j}]$ . For every value of  $j$ ,  $\pi_{> j}$  is non-empty and greater than 1, and so  $\mu[\varepsilon, \pi_{> j}] = 0$  for all  $j$ , and hence every term in the sum is zero. If  $\pi = 1 \oplus (\oplus^r \alpha) \oplus \tau$  or  $\pi = 1 \oplus (\oplus^r \alpha) \oplus \tau \oplus 1$ , then we use Proposition 5, which reduces to one of the previous cases.  $\square$

We now turn to identifying pairs and quartets of permutations that make no net contribution to the Möbius sum. We start by showing that if  $\sigma$  and  $\alpha$  are indecomposable, and  $r \geq 1$ , and with  $\pi \in \mathcal{F}_\oplus(\oplus^r \alpha)$ , then  $\mu[\sigma, \pi]$  and  $\mu[\sigma, \alpha]$  have the same magnitude.

**Lemma 9.** *Let  $\pi \in \mathcal{F}_\oplus(\oplus^r \alpha)$ , where  $r \geq 1$  and  $\alpha > 1$  is sum indecomposable. Let  $\sigma$  be a sum indecomposable permutation. Then*

$$\mu[\sigma, \pi] = \begin{cases} \mu[\sigma, \alpha] & \text{if } \pi = \oplus^r \alpha \text{ or } 1 \oplus (\oplus^r \alpha) \oplus 1, \\ -\mu[\sigma, \alpha] & \text{if } \pi = 1 \oplus (\oplus^r \alpha) \text{ or } (\oplus^r \alpha) \oplus 1. \end{cases}$$

As a consequence, if  $\mathcal{F}_\oplus(\oplus^r \alpha) \subseteq [\sigma, \pi]$ , then  $\mathcal{F}_\oplus(\oplus^r \alpha)$  makes no net contribution to  $\mu[\sigma, \pi]$ .

*Proof.* If  $\pi = \oplus^r \alpha$  or  $\pi = (\oplus^r \alpha) \oplus 1$ , then this is immediate from Corollary 7. If  $\pi = 1 \oplus (\oplus^r \alpha)$  or  $\pi = 1 \oplus (\oplus^r \alpha) \oplus 1$ , then we use Proposition 5.

For the net contribution of  $\mathcal{F}_\oplus(\oplus^r \alpha)$ ,  $\sum_{\lambda \in \mathcal{F}_\oplus(\oplus^r \alpha)} \mu[\sigma, \lambda] = 0$ . □

We now have a lemma that adds a further restriction to the permutations that have a non-zero contribution to the Möbius sum.

**Lemma 10.** *If  $\sigma \leq \pi$ , and  $\alpha \in [\sigma, \pi]$  is sum indecomposable, and  $r$  is the smallest integer such that  $1 \oplus (\oplus^r \alpha) \oplus 1 \not\leq \pi$ , then  $\mathcal{F}_\oplus(\oplus^k \alpha) \subseteq [\sigma, \pi]$  for all  $k \in [1, r]$ .*

*Proof.* For any  $k < r$ ,  $\sigma \leq \oplus^k \alpha < 1 \oplus (\oplus^k \alpha) \oplus 1 \leq \pi$ . Note that by Lemma 9 the net contribution of the family  $\mathcal{F}_\oplus(\oplus^k \alpha)$  to  $\mu[\sigma, \pi]$  is zero. □

**Observation 11.** *Using the same terminology as Lemma 10, if  $k > r + 1$  then we must have  $\oplus^k \alpha \not\leq \pi$ . As a consequence, for each indecomposable  $\alpha \in [\sigma, \pi]$ , the only families of  $\alpha$  that can have a non-zero net contribution to  $\mu[\sigma, \pi]$  are  $\mathcal{F}_\oplus(\oplus^r \alpha)$  and  $\mathcal{F}_\oplus(\oplus^{r+1} \alpha)$ .*

We now eliminate two specific permutations from the Möbius sum.

**Lemma 12.** *If  $\pi$  is any permutation with  $|\pi| > 3$  apart from the identity permutation and its reverse, and  $\sigma$  is sum indecomposable, then the permutations 1 and  $1 \oplus 1$  make no net contribution to the Möbius sum  $\mu[\sigma, \pi]$ .*

*Proof.* If  $\sigma = 1$ , then the interval contains both 1 and  $1 \oplus 1$ . Since  $\mu[1, 1] = 1$  and  $\mu[1, 12] = -1$ , there is no net contribution to  $\mu[\sigma, \pi]$ . If  $\sigma > 1$ , then  $\sigma \neq 12$ , and so neither 1 nor 12 is in the interval.  $\square$

Before we present the main theorem for this section, we formally define the weight function and the contributing set. Let  $\alpha$  be a sum indecomposable permutation. The weight function,  $W(\sigma, \alpha, \pi)$ , is defined as

$$W(\sigma, \alpha, \pi) = \begin{cases} 1 & \text{If } \begin{cases} \sigma \leq \oplus^r \alpha \leq \pi \text{ and} \\ 1 \oplus (\oplus^r \alpha) \not\leq \pi \text{ and} \\ (\oplus^r \alpha) \oplus 1 \not\leq \pi, \end{cases} \\ -1 & \text{If } \begin{cases} \sigma \leq \oplus^r \alpha \leq \pi \text{ and} \\ 1 \oplus (\oplus^r \alpha) \leq \pi \text{ and} \\ (\oplus^r \alpha) \oplus 1 \leq \pi \text{ and} \\ \oplus^{r+1} \alpha \not\leq \pi, \end{cases} \\ 0 & \text{Otherwise,} \end{cases} \quad (4.3)$$

where  $r$  is the smallest integer such that  $1 \oplus (\oplus^r \alpha) \oplus 1 \not\leq \pi$ .

The contributing set  $\mathfrak{C}_{\sigma, \pi}$  is defined as

$$\mathfrak{C}_{\sigma, \pi} = \left\{ \begin{array}{l} \alpha \in [\sigma, \pi), \\ \alpha : \alpha \text{ is sum indecomposable, and} \\ W(\sigma, \alpha, \pi) \neq 0 \end{array} \right\}.$$

We have one last lemma before we move on to the main theorem.

**Lemma 13.** *If  $\sigma$  and  $\alpha$  are sum indecomposable, then for any permutation  $\pi$ ,  $\mu[\sigma, \alpha]W(\sigma, \alpha, \pi)$  gives the contribution of the set of families  $\mathcal{F}_{\oplus}(\oplus^r \alpha)$  to the Möbius sum, where  $r$  is any positive integer.*

$1 \oplus (\oplus^r \alpha)$	$(\oplus^r \alpha) \oplus 1$	$\oplus^{r+1} \alpha$	Möbius contribution
$\leq \pi$	$\leq \pi$	$\leq \pi$	0
$\leq \pi$	$\leq \pi$	$\not\leq \pi$	$-\mu[\sigma, \alpha]$
$\leq \pi$	$\not\leq \pi$	$\not\leq \pi$	0
$\not\leq \pi$	$\leq \pi$	$\not\leq \pi$	0
$\not\leq \pi$	$\not\leq \pi$	$\not\leq \pi$	$\mu[\sigma, \alpha]$

Table 4.1: Möbius contribution from family members.

*Proof.* By Observation 11, we only need consider the contribution made by  $\oplus^r \alpha$  and  $\oplus^{r+1} \alpha$ , where  $r$  is the smallest integer such that  $1 \oplus (\oplus^r \alpha) \oplus 1 \not\leq \pi$ .

If  $\sigma \not\leq \oplus^r \alpha$ , or  $\oplus^r \alpha \not\leq \pi$ , then  $\mathcal{F}_\oplus(\oplus^r \alpha)$  makes no net contribution to the Möbius sum. Now assume that  $\sigma \leq \oplus^r \alpha \leq \pi$ . First, we can see that if  $1 \oplus (\oplus^r \alpha) \not\leq \pi$ , or  $(\oplus^r \alpha) \oplus 1 \not\leq \pi$  then  $\oplus^{r+1} \alpha \not\leq \pi$ . We can also see that if  $1 \oplus (\oplus^r \alpha) \oplus 1 \not\leq \pi$  then  $1 \oplus (\oplus^{r+1} \alpha) \not\leq \pi$  and  $\oplus^{r+1} \alpha \not\leq \pi$ . The possibilities remaining are itemised in Table 4.1, where the Möbius contribution is determined by applying Lemma 9. We can see that in every case  $W(\sigma, \alpha, \pi)$  provides the correct weight for the Möbius function  $\mu[\sigma, \alpha]$ .  $\square$

We are now in a position to present the main theorem for this section.

**Theorem 14.** *If  $\sigma$  is a sum indecomposable permutation, and  $|\pi| > 3$ , then*

$$\mu[\sigma, \pi] = - \sum_{\alpha \in \mathfrak{C}_{\sigma, \pi}} \mu[\sigma, \alpha] W(\sigma, \alpha, \pi).$$

*Proof.* Let  $\alpha \leq \pi$  be an indecomposable permutation.

Using Lemmas 8 and 12 we can see that any permutations not in the set  $\mathcal{F}_\oplus(\oplus^r \alpha)$  can be excluded from  $\mathfrak{C}_{\sigma, \pi}$ , as these permutations make no net contribution to the Möbius sum.

For every  $\alpha$ , by Lemma 13,  $\mu[\sigma, \alpha] W(\sigma, \alpha, \pi)$  provides the contribution to the Möbius sum of all families  $\mathcal{F}_\oplus(\oplus^r \alpha)$ , where  $r$  is a positive integer.  $\square$

Theorem 14 reduces the number of permutations that need to be considered as part of the Möbius sum. We can see that the largest permutation in  $\mathfrak{C}_{\sigma,\pi}$  must have length less than  $|\pi|$ , and so we can apply Theorem 14 recursively to the permutations in  $\mathfrak{C}_{\sigma,\pi}$  to determine their Möbius values. In this recursion, if we are attempting to determine  $\mu[\sigma, \lambda]$ , we can stop if  $|\sigma| = |\lambda|$  or  $|\sigma| = |\lambda| - 1$ , as in these cases  $\mu[\sigma, \lambda]$  is  $+1$  and  $-1$  respectively.

## 4.5 Increasing oscillations

We now move on to increasing oscillations. Given an indecomposable permutation  $\sigma$ , and an increasing oscillation  $\pi$ , our aim in this section is to describe  $\mathfrak{C}_{\sigma,\pi}$  in precise terms. We will find a sum for the Möbius function,  $\mu[\sigma, \pi]$ , which only requires the evaluation of simple inequalities.

If  $\pi$  is an increasing oscillation with length less than 4, then  $\mu[\sigma, \pi]$  is trivial to determine for any  $\sigma$ . For the remainder of this section we assume that  $\pi$  has length at least 4.

We partition the set of increasing oscillations with length greater than 1 into five disjoint subsets. These subsets are  $\{21\}$ ,  $\{\odot^{k+1}21\}$ ,  $\{1 \odot (\odot^k 21)\}$ ,  $\{(\odot^k 21) \odot 1\}$ , and  $\{1 \odot (\odot^k 21) \odot 1\}$ , where  $k$  is a positive integer. If two increasing oscillations are in the same subset, then we say that they have the same *shape*.

We now determine what permutations contained in an increasing oscillation have a non-zero contribution to the Möbius sum.

**Lemma 15.** *Let  $\pi$  be an increasing oscillation, and let  $\sigma \leq \pi$  be sum indecomposable. Let  $S$  be the subset of the permutations in the interval  $[\sigma, \pi)$  that can be written in the form  $\mathcal{F}_{\oplus}(\oplus^r \mathcal{F}_{\odot}(\odot^k 21))$  for some  $k, r \geq 1$ . If  $\lambda \in [\sigma, \pi)$ , and  $\lambda \notin S$ , then  $\mu[\sigma, \lambda] = 0$ .*

We note here that  $\mathcal{F}_{\odot}(\odot^k 21)$  is a set containing only increasing oscillations.

*Proof.* We start by showing that if  $\pi$  is an increasing oscillation, and  $\lambda = \lambda_1 \oplus \dots \oplus \lambda_m \leq \pi$ , where each  $\lambda_i$  is sum indecomposable, then every  $\lambda_i$  is an increasing oscillation. This is trivially true if  $\lambda$  is itself an increasing oscillation, thus it is sufficient to show that if  $\lambda$  is an increasing oscillation, then deleting a single point results in either an increasing oscillation, or a permutation that is the sum of two increasing oscillations.

If  $k = 1$ , then we can see that deleting a single point results in a permutation with the required characteristic.

Now assume that  $k > 1$ . Let  $\lambda = 1 \otimes (\otimes^k 21)$ . Deleting the leftmost point gives  $\otimes^k 21$ , and deleting the rightmost point gives  $1 \otimes (\otimes^{k-1} 21) \oplus 1$ . Deleting the second point gives  $21 \oplus (\otimes^{k-1} 21)$ , and deleting the last-but-one point gives  $1 \otimes (\otimes^{k-1} 21) \otimes 1$ . Deleting any even point  $2t$  except the second or second-to-last results in  $(1 \otimes (\otimes^{t-1} 21) \otimes 1) \oplus ((\otimes^{k-t} 21))$ . Finally, deleting any odd point  $2t+1$  apart from the first or last results in  $(1 \otimes (\otimes^{t-1} 21)) \oplus (1 \otimes (\otimes^{k-t} 21))$ . Thus if  $\lambda = 1 \otimes (\otimes^k 21)$ , then deleting a single point from  $\lambda$  results in either an increasing oscillation, or a permutation that is the sum of two increasing oscillations.

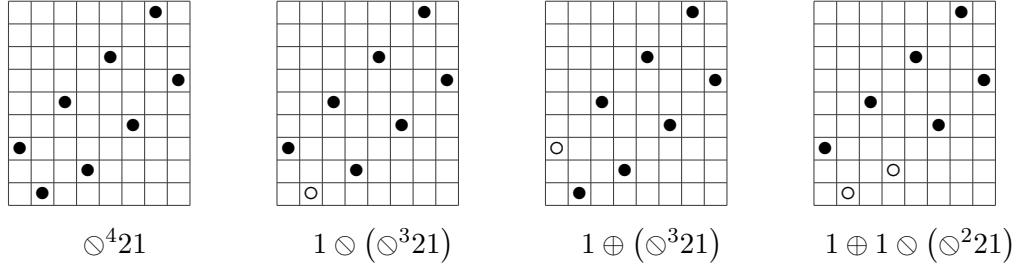
A similar argument applies to the other three cases, which we omit for brevity.

To complete the proof, we now see that by Lemma 12, we can ignore  $\lambda = 1$  and  $\lambda = 1 \oplus 1$ . If  $\lambda = \lambda_1 \oplus \lambda_2 \oplus \dots \oplus \lambda_m \leq \pi$ , then by the argument above, every  $\lambda_i$  is an increasing oscillation. Applying Lemma 8 completes the proof.  $\square$

Following Observation 11, it is clear that, if  $\alpha \in \mathcal{F}_\otimes(\otimes^k 21)$ , then for any family  $\mathcal{F}_\oplus(\oplus^r \alpha)$ , we only need consider the cases  $\oplus^r \alpha$  and  $\oplus^{r+1} \alpha$  where  $r$  is the smallest integer such that  $1 \oplus (\oplus^r \alpha) \oplus 1 \not\leq \pi$ .

Given some  $\pi = W_n$  or  $M_n$ , we will find inequalities that relate  $n$ ,  $r$  and  $k$  and the shape of  $\alpha$  that will allow us to find the values that contribute to the Möbius sum. We know from Lemma 15 the shape of the permutations that contribute to the Möbius sum. For each of the four types of increasing oscillation ( $W_{2n}$ ,  $W_{2n-1}$ ,  $M_{2n}$  and  $M_{2n-1}$ ), we can examine how each shape can be embedded so that the unused points at the start of the increasing oscillation are minimised. Figure 4.3 shows examples of embeddings into  $W_{2n}$ . This gives us an inequality relating to the start of the embedding. Similarly, we can find inequalities for the end of the embedding. We can also find inequalities that relate to the interior (when  $r > 1$ ), and Figures 4.4 and 4.5 show examples of this. We can use these inequalities to determine what values of  $k$  will allow the shape to be embedded. For each allowable value of  $k$ , we can then determine the maximum value of  $r$  such that  $1 \oplus (\oplus^r \alpha) \oplus 1 \not\leq \pi$ . This then means that, by evaluating inequalities alone, we can identify the specific permutations that could contribute to the Möbius sum.

We first have two lemmas that examine inequalities at the start and end of an

Figure 4.3: Embedding the start of  $\alpha$  in  $W_{2n}$ .

embedding.

**Lemma 16.** *If  $\pi$  is an increasing oscillation, and  $\alpha \leq \pi$  is sum indecomposable, then in any embedding of an element  $\lambda$  of  $\mathcal{F}_\oplus(\oplus^r \alpha)$  into  $\pi$ , the minimum number of unused points at the start of  $\pi$  depends on the start of  $\lambda$ , and on  $\pi$ , and is as shown below:*

Start of $\lambda$	$\pi = W_{2n}$	$\pi = W_{2n-1}$	$\pi = M_{2n}$	$\pi = M_{2n-1}$
$21 \dots$	0	0	0	0
$\otimes^{k+1} 21 \dots$	0	0	1	1
$1 \otimes (\otimes^k 21) \dots$	1	1	0	0
$1 \oplus 21 \dots$	1	1	1	1
$1 \oplus (\otimes^{k+1} 21) \dots$	1	1	2	2
$1 \oplus 1 \otimes (\otimes^k 21) \dots$	2	2	1	1

*Proof.* It is clear that if we minimise the number of points at the start of an embedding, then the number of unused points depends on  $\pi$ , and the start of  $\alpha$ . The values in Lemma 16 are found by considering each of the possibilities. We illustrate some of these cases in Figure 4.3. □

**Lemma 17.** *If  $\pi$  is an increasing oscillation, and  $\alpha \leq \pi$  is sum indecomposable, then in any embedding of an element  $\lambda$  of  $\mathcal{F}_\oplus(\oplus^r \alpha)$  into  $\pi$ , the minimum number of unused points at the end of  $\pi$  depends on the end of  $\lambda$ , and on  $\pi$ , and is as shown below:*



<i>End of <math>\lambda</math></i>	$\pi = W_{2n}$	$\pi = W_{2n-1}$	$\pi = M_{2n}$	$\pi = M_{2n-1}$
$\dots 21$	0	0	0	0
$\dots \otimes^{k+1} 21$	0	1	1	0
$\dots (\otimes^k 21) \otimes 1$	1	0	0	1
$\dots 21 \oplus 1$	1	1	1	1
$\dots (\otimes^{k+1} 21) \oplus 1$	1	2	2	1
$\dots (\otimes^k 21) \otimes 1 \oplus 1$	2	1	1	2

*Proof.* We examine all the possibilities as we did in Lemma 16.  $\square$

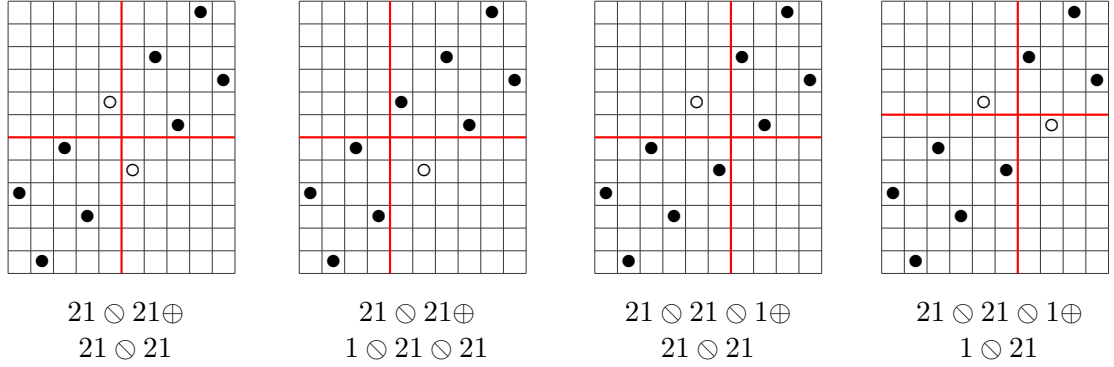
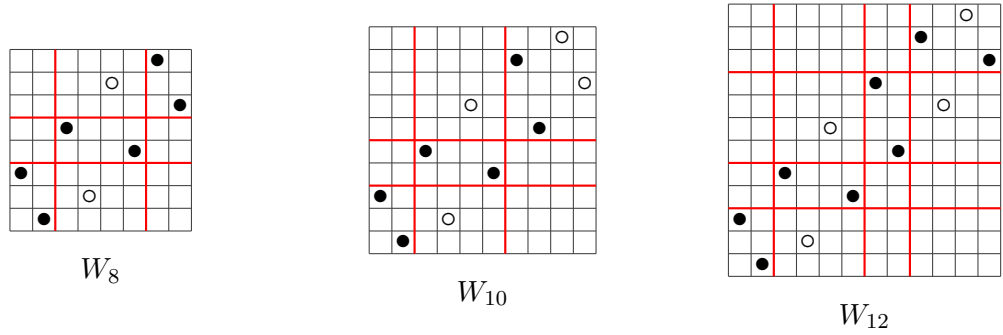
We now consider how closely copies of some sum indecomposable  $\alpha$  can be embedded into  $\pi$ . This leads to two inequalities that relate  $\alpha$ ,  $\pi$  and the maximum number of copies of  $\alpha$  that can be embedded in  $\pi$ . Where  $\alpha \neq 21$ , the shape of  $\alpha$  fixes the way the two copies can be embedded in an increasing oscillation. If  $\alpha = 21$ , then we will see that there are choices for the embedding.

**Lemma 18.** *If  $\pi$  is an increasing oscillation, and  $\alpha \neq 21$ , and  $\alpha \leq \pi$  is sum indecomposable, then in any embedding of  $\oplus^r \alpha$  into  $\pi$ , the minimum number of points between the start and end of  $\oplus^r \alpha$  depends on  $\alpha$ , and is as shown below:*

<i>Shape of <math>\alpha</math></i>	<i>Points in <math>\oplus^r \alpha</math></i>	<i>Unused points</i>	<i>Minimum points</i>
$\otimes^{k+1} 21$	$2kr$	$2r - 2$	$2kr + 2r - 2$
$1 \otimes (\otimes^k 21)$	$2kr + r$	$r - 1$	$2kr + 2r - 1$
$(\otimes^k 21) \otimes 1$	$2kr + r$	$r - 1$	$2kr + 2r - 1$
$1 \otimes (\otimes^k 21) \otimes 1$	$2kr + 2r$	$2r - 2$	$2kr + 4r - 2$

*Proof.* If  $r = 1$ , then there are no unused points, and so the minimum number of points depends solely on the points in  $\alpha$ , and the table reflects this.

Assume now that  $r > 1$ . If  $\alpha \neq 21$ , then we can see that the interleave fixes the layout of each copy of  $\alpha$ , so we simply pack each copy as close as possible. This packing clearly depends on the start and end of  $\alpha$ , and it is simple to examine the four possibilities. Examples are shown in Figure 4.4.  $\square$

Figure 4.4: Packing  $\alpha$  as close as possible when  $\alpha \neq 21$ .Figure 4.5: Examples of unused points when embedding  $\oplus^r 21$ .

We now turn to the case where  $\alpha = 21$ . This is more complex than the previous cases. We can see that there must be at least one point between each copy of  $\alpha$ . We can insert each copy of 21 in two ways, one where the points are horizontally adjacent, and one where the points are vertically adjacent. These alternatives can be seen in Figure 4.5. Alternating these means that there will be exactly one point between each copy of  $\alpha$ , so this embedding minimises the number of points between the start and end of  $\oplus^r \alpha$ . The complication in this case relates to how we start and end the embedding. We illustrate this by showing, in Figure 4.5, maximal embeddings where we are embedding into  $W_8$ ,  $W_{10}$  and  $W_{12}$ . A detailed examination of each possible case gives us our second inequality.

**Lemma 19.** *If  $\pi$  is an increasing oscillation, and  $\alpha = 21$  then for  $\oplus^r \alpha$  to be contained in  $\pi$  we must have  $3r - 1 \leq 2n$  for  $\pi \in \{W_{2n}, M_{2n}\}$ , and  $3r \leq 2n$  for*

$$\pi \in \{W_{2n-1}, M_{2n-1}\}.$$

*Proof.* In every case we start by embedding the first 21 into the first two elements of the permutation. Thereafter, we embed each successive 21 as close as possible to the preceding 21. The minimum number of elements to embed  $r$  copies of 21 will be  $2r$  elements to hold the points of the 21s, and  $r - 1$  intermediate empty elements. For  $W_{2n}$  and  $M_{2n}$ , this then gives  $3r - 1 \leq 2n$ , and for  $W_{2n-1}$  and  $M_{2n-1}$  we obtain  $3r - 1 \leq 2n - 1$ .  $\square$

We now have a complete understanding of the number of points required to embed any permutation that contributes to the Möbius sum into an increasing oscillation. The following Lemma summarises the situation.

**Lemma 20.** *If  $\pi$  is an increasing oscillation, and  $\alpha \in \mathcal{F}_{\odot}(\odot^k 21) \leq \pi$  (so  $\alpha$  is sum indecomposable), then for  $\oplus^r \alpha$  to be contained in  $\pi$ , the inequality in the table below must be satisfied, where  $k \geq 1$ .*

$\pi$	Shape of $\alpha$	Inequality
$W_{2n}, M_{2n}$	21	$3r - 1 \leq 2n$
$W_{2n-1}, M_{2n-1}$	21	$3r \leq 2n$
$W_{2n}$	$\odot^{k+1} 21$	$2kr + 2r - 2 \leq 2n$
$W_{2n-1}$	$1 \odot (\odot^k 21)$	$2kr + 2r + 2 \leq 2n$
$M_{2n-1}$	$(\odot^k 21) \odot 1$	$2kr + 2r + 2 \leq 2n$
$M_{2n}$	$1 \odot (\odot^k 21) \odot 1$	$2kr + 4r - 2 \leq 2n$
$W_{2n}, W_{2n-1}, M_{2n-1}$	$1 \odot (\odot^k 21) \odot 1$	$2kr + 4r \leq 2n$
<i>All other cases</i>		$2kr + 2r \leq 2n$

*Proof.* We apply Lemmas 16, 17, 18 and 19 to the possibilities for  $\pi$  and  $\alpha$ .  $\square$

As a consequence of Lemmas 16 and 17 we can define a relationship between the minimum number of points required to embed some  $\oplus^r \alpha$ , and the minimum number of points required to embed  $1 \oplus (\oplus^r \alpha)$ ,  $\oplus^r \alpha \oplus 1$  and  $1 \oplus (\oplus^r \alpha) \oplus 1$ .

**Corollary 21.** *If  $\pi$  is an increasing oscillation, and  $\alpha \leq \pi$  is sum indecomposable and if the minimum number of points required to embed  $\oplus^r \alpha$  into  $\pi$  is  $C$ , then the minimum number of points required to embed  $1 \oplus (\oplus^r \alpha)$  into  $\pi$  is  $C+2$ , the minimum number of points required to embed  $\oplus^r \alpha \oplus 1$  into  $\pi$  is  $C+2$ , and the minimum number of points required to embed  $1 \oplus (\oplus^r \alpha) \oplus 1$  into  $\pi$  is  $C+4$ .*

*Proof.* We can see from Lemmas 16 and 17 that adding  $1 \oplus$  at the start of a permutation increases the number of points required by two – one for the new point, and one that is unused. Similarly, adding  $\oplus 1$  at the end increases the points required by two.  $\square$

Lemma 20 gives us inequalities that any  $\oplus^r \alpha$  must satisfy to ensure that  $\oplus^r \alpha \leq \pi$ . Further, Corollary 21 gives us inequalities that, for a given  $\oplus^r \alpha$  allow us to determine if  $1 \oplus (\oplus^r \alpha) \leq \pi$ ,  $(\oplus^r \alpha) \oplus 1 \leq \pi$  and  $1 \oplus (\oplus^r \alpha) \oplus 1 \leq \pi$ . We can therefore determine what values of  $r$  and  $k$  will result in  $\mathcal{F}_{\oplus}(\oplus^r \alpha)$  contributing to the Möbius function. We now consider inequalities that relate  $\sigma$  and  $\alpha$ , so that we can determine if  $\sigma \leq \alpha$  using an inequality.

**Lemma 22.** *If  $\sigma > 1$  is an increasing oscillation, and  $\alpha \in \mathcal{F}_{\odot}(\odot^k 21)$  for some  $k$ , then for  $\sigma$  to be contained in  $\alpha$  the inequality in the table below must be satisfied, where  $k \geq 1$ .*

$\sigma$	Shape of $\alpha$	Inequality
$W_{2n-1}, M_{2n}, M_{2n-1}$	21	False
$W_{2n-1}$	$(\odot^k 21) \odot 1$	$k \geq n-1$
$M_{2n-1}$	$1 \odot (\odot^k 21)$	$k \geq n-1$
$W_{2n-1}, M_{2n}, M_{2n-1}$	$1 \odot (\odot^k 21) \odot 1$	$k \geq n-1$
$M_{2n}$	$\odot^{k+1} 21$	$k \geq n+1$
All other cases		$k \geq n$

*Proof.* We examine all possible cases.  $\square$

We are now nearly ready to present the main theorem for this section. Informally, for each possible shape of permutation  $\alpha$ , we will first find the minimum and maximum

values of  $k$  such that  $\sigma \leq \alpha \leq \pi$ , as any other values of  $k$  result in  $\alpha$  being outside the interval. For each  $\alpha$  and each  $k$ , we then determine the minimum value of  $r$  such that  $1 \oplus (\oplus^r \alpha) \oplus 1 \not\leq \pi$ . We can then use this value of  $r$  (assuming it is non-zero) to determine the weight to be applied to  $\mu[\sigma, \alpha]$ . The set of  $\alpha$ 's with a non-zero weight is a contributing set  $\mathfrak{C}_{\sigma, \pi}$ . At this point we can substitute a value for any  $\mu[\sigma, \alpha]$  where  $|\sigma| \leq |\alpha| - 1$ . We then use the same process recursively to determine the contributing set for the remaining elements of  $\mathfrak{C}_{\sigma, \pi}$ .

We first define some supporting functions. Let  $\text{RawMinK}(\sigma, \alpha)$  be the minimum value of  $k$  that satisfies the inequality in Lemma 22. For the first inequality, which is always false, we set  $k = |\pi|$ , as this will force the sum, defined later in Theorem 23, to be empty.

Let  $\text{MinK}(\sigma, \alpha)$  be defined as

$$\text{MinK}(\sigma, \alpha) = \begin{cases} 1 & \text{If } \sigma = 1 \text{ and } \alpha \neq \odot^{k+1} 21, \\ 2 & \text{If } \sigma = 1 \text{ and } \alpha = \odot^{k+1} 21, \\ \text{RawMinK}(\sigma, \alpha) & \text{otherwise.} \end{cases}$$

Observe that for any  $k < \text{MinK}(\sigma, \alpha)$ , we have  $\alpha < \sigma$ , and so  $\mathcal{F}_{\oplus}(\oplus^k \alpha)$  makes no net contribution to the Möbius sum.

Let  $\text{MaxK}(\alpha, \pi)$  be defined as the maximum value of  $k$  that satisfies the inequality in Lemma 20, if the shape of  $\alpha$  and the shape of  $\pi$  are different; and one less than the maximum value of  $k$  that satisfies the inequality if the shape of  $\alpha$  and the shape of  $\pi$  are the same. For the first two inequalities, which do not involve  $k$ , we set  $\text{MaxK}(\alpha, \pi) = 1$  if the inequality is satisfied, and  $\text{MaxK}(\alpha, \pi) = 0$  if not. Observe here that for any  $k > \text{MaxK}(\alpha, \pi)$  we have  $\alpha \not\leq \pi$ , and so  $\mathcal{F}_{\oplus}(\oplus^k \alpha)$  makes no contribution to the Möbius sum.

We define the weight function for increasing oscillations,  $W_{io}(\sigma, \alpha, \pi)$ , as

$$W_{io}(\sigma, \alpha, \pi) = \begin{cases} 1 & \text{If } (\oplus^r \alpha) \oplus 1 \not\leq \pi, \\ -1 & \text{If } (\oplus^r \alpha) \oplus 1 \leq \pi \text{ and } \oplus^{r+1} \alpha \not\leq \pi, \\ 0 & \text{Otherwise,} \end{cases}$$

where  $r$  is the smallest integer such that  $1 \oplus (\oplus^r \alpha) \oplus 1 \not\leq \pi$ . These conditions are simpler than those given in the weight function (4.3) for Theorem 14 as, by Corollary 21, if  $(\oplus^r \alpha) \oplus 1 \not\leq \pi$  then  $1 \oplus (\oplus^r \alpha) \not\leq \pi$  and vice-versa. Furthermore, we will see that this weight function is only used when  $\sigma \leq \oplus^r \alpha \leq \pi$ .

We are now in a position to state our main theorem for this section. In this theorem, we consider the contribution to the Möbius sum of each possible shape of some sum indecomposable  $\alpha$ . There are five possible shapes, and, given that the expression for each shape is identical, we abuse notation slightly by writing our theorem as a sum over the shapes, thus the first sum in Theorem 23 is over the possible shapes of  $\alpha$ , where four of the shapes have a parameter  $k$ . For each shape, the limits on the interior sum determine the minimum and maximum values of  $k$ , using the summation variable  $v$ . We use the notation  $\alpha_v$  to represent the actual permutation that has the shape  $\alpha$ , where the parameter  $k$  has been set to the value of  $v$ . As an example, if  $\alpha = 1 \odot (\odot^k 21)$ , and  $v = 2$ , then  $\alpha_v = 1 \odot (\odot^2 21) = 24153$ .

**Theorem 23.** *Let  $\pi$  be an increasing oscillation, and let  $\sigma \leq \pi$  be sum indecomposable. Then*

$$\mu[\sigma, \pi] = \sum_{\alpha \in \mathcal{S}} \sum_{v=\text{MinK}(\sigma, \alpha)}^{\text{MaxK}(\alpha, \pi)} \mu[\sigma, \alpha_v] W_{io}(\sigma, \alpha_v, \pi)$$

where the first sum is over the possible shapes of a sum indecomposable permutation contained in an increasing oscillation, so  $\mathcal{S} = \{21, \odot^{k+1} 21, 1 \odot (\odot^k 21), (\odot^k 21) \odot 1, 1 \odot (\odot^k 21) \odot 1\}$ .

*Proof.* By Lemma 15 the only sum-decomposable permutations contained in an increasing oscillation that contribute to the Möbius sum are  $\mathcal{F}_{\oplus}(\oplus^r \alpha)$ , where  $\alpha \in \mathcal{S}$ .

If we set  $r = 1$ , then for each  $\alpha$  in  $\mathcal{S}$  Lemma 22 provides the smallest value of  $k$  such that  $\sigma \leq \alpha$ . If there is no such value of  $k$ , then we use  $|\pi|$ , as the maximum value of  $k$  must be smaller than this, and so the sum is empty.

Again setting  $r = 1$ , for each  $\alpha$  in  $\mathcal{S}$  Lemma 20 provides the maximum value of  $k$  such that  $\alpha \leq \pi$ . If there is no value of  $k$  that satisfies the inequality, then we set  $\text{MaxK}(\alpha, \pi) = 0$ , thus forcing the sum to be empty.

Thus the permutations  $\alpha_v$  in the sum

$$\sum_{\alpha \in \mathcal{S}} \sum_{v=\text{MinK}(\sigma, \alpha)}^{\text{MaxK}(\alpha, \pi)}$$

are those that could contribute to the Möbius sum, and for any  $\alpha_v$  not included in the sum,  $\mathcal{F}_{\oplus}(\oplus^r \alpha_v)$  has a zero contribution to the Möbius sum for any  $r$ .

Further, we can see from the construction method that any  $\alpha_v$  included in the sum has  $\sigma \leq \oplus^r \alpha_v \leq \pi$  for at least one value of  $r$ , as if this was not the case, then we would have  $\text{MinK}(\sigma, \alpha) > \text{MaxK}(\alpha, \pi)$ , and so the sum would be empty.

We have therefore shown that the  $\alpha_v$ -s included in the sum form a contributing set, and we could therefore set  $\mathfrak{C}_{\sigma, \pi}$  to be those  $\alpha_v$ -s, and use Theorem 14. We now show that the increasing oscillation weight function  $W_{io}(\sigma, \alpha, \pi)$  is equivalent to  $W(\sigma, \alpha, \pi)$  as defined in the general case.

By Corollary 21, if  $(\oplus^r \alpha) \oplus 1 \not\leq \pi$  then  $1 \oplus (\oplus^r \alpha) \not\leq \pi$  and vice-versa, and so the condition for  $(\oplus^r \alpha) \oplus 1$  also covers  $1 \oplus (\oplus^r \alpha)$ . As discussed above, we know that there is at least one value of  $r$  such that  $\sigma \leq \oplus^r \alpha \leq \pi$ , and so  $W_{io}(\sigma, \alpha, \pi)$  does not need to include this condition. Thus the increasing oscillation weight function  $W_{io}(\sigma, \alpha, \pi)$  is equivalent to  $W(\sigma, \alpha, \pi)$  as defined in the general case.  $\square$

### 4.5.1 Example of Theorem 23

As an example of Theorem 23 in action, we show how to determine

$$\mu[3142, 315274968] = \mu[\odot^2 21, \odot^4 21 \odot 1].$$

We start by considering each possible shape of  $\alpha$ , setting  $r = 1$ , and then using the inequalities in Lemmas 20 and 22 to determine the minimum and maximum values of  $k$ . This gives us

Shape of $\alpha$	Minimum $k$	Maximum $k$
21	1	1
$\otimes^k 21$	2	4
$1 \otimes (\otimes^k 21)$	2	3
$(\otimes^k 21) \otimes 1$	2	3
$1 \otimes (\otimes^k 21) \otimes 1$	2	3

For each shape of  $\alpha$ , and each value of  $k$ , we then use the inequalities in Lemma 20 to determine the minimum value of  $r$  such that  $1 \oplus (\oplus^r \alpha) \oplus 1 \not\leq \pi$ , and we then calculate the weight using this value of  $r$ . This gives

$\alpha$	$r$	Weight
21	No possibilities	
$\otimes^2 21$	2	1
$\otimes^3 21$	1	-1
$\otimes^4 21$	1	-1
$1 \otimes (\otimes^2 21)$	1	-1
$1 \otimes (\otimes^3 21)$	1	-1
$(\otimes^2 21) \otimes 1$	2	1
$(\otimes^3 21) \otimes 1$	1	-1
$1 \otimes (\otimes^2 21) \otimes 1$	1	-1
$1 \otimes (\otimes^3 21) \otimes 1$	1	-1

This leads to the following initial expression:

$$\begin{aligned}
\mu[\otimes^2 21, \otimes^4 21 \otimes 1] = & \mu[\otimes^2 21, \otimes^2 21] - \mu[\otimes^2 21, \otimes^3 21] - \mu[\otimes^2 21, \otimes^4 21] \\
& - \mu[\otimes^2 21, 1 \otimes (\otimes^2 21)] - \mu[\otimes^2 21, 1 \otimes (\otimes^3 21)] \\
& + \mu[\otimes^2 21, \otimes^2 21 \otimes 1] - \mu[\otimes^2 21, \otimes^3 21 \otimes 1] \\
& - \mu[\otimes^2 21, 1 \otimes (\otimes^2 21) \otimes 1] - \mu[\otimes^2 21, 1 \otimes (\otimes^3 21) \otimes 1]
\end{aligned}$$



We know that  $\mu[\otimes^2 21, \otimes^2 21] = 1$ , and that

$$\mu[\otimes^2 21, 1 \otimes (\otimes^2 21)] = \mu[\otimes^2 21, \otimes^2 21 \otimes 1] = -1.$$

Applying Theorem 23 recursively to the other intervals eventually yields

$$\mu[\otimes^2 21, \otimes^4 21 \otimes 1] = -6.$$

## 4.6 Concluding remarks

The results in [18] provide two recurrences to handle the case where  $\pi$  is decomposable. This work handles the case where  $\sigma$  is indecomposable. It overlaps with [18] when  $\sigma$  is indecomposable and  $\pi$  is decomposable. This leaves the case where  $\sigma$  is decomposable and  $\pi$  is indecomposable for further investigation.

We can see that by symmetry  $\mu[\sigma, W_n] = \mu[\sigma^{-1}, M_n]$ . If we consider the value of the principal Möbius function,  $\mu[1, \pi]$ , where  $\pi$  is either  $W_n$  or  $M_n$ , then it is simple to show that the absolute value of the principal Möbius function is bounded above by  $2^n$ . The weight function for increasing oscillations can be  $\pm 1$ , and we can see no obvious reason why there should not be two distinct values,  $i$  and  $j$ , with the same parity, such that the signs of  $\mu[1, W_i]$  and  $\mu[1, W_j]$  were different. We have experimental evidence, based on the values of  $W_n$  and  $M_n$  for  $n = 1 \dots 2,000,000$  that suggests that  $\mu[1, W_{2n}] < 0$ , and that  $\mu[1, W_{2n-1}] > 0$ .

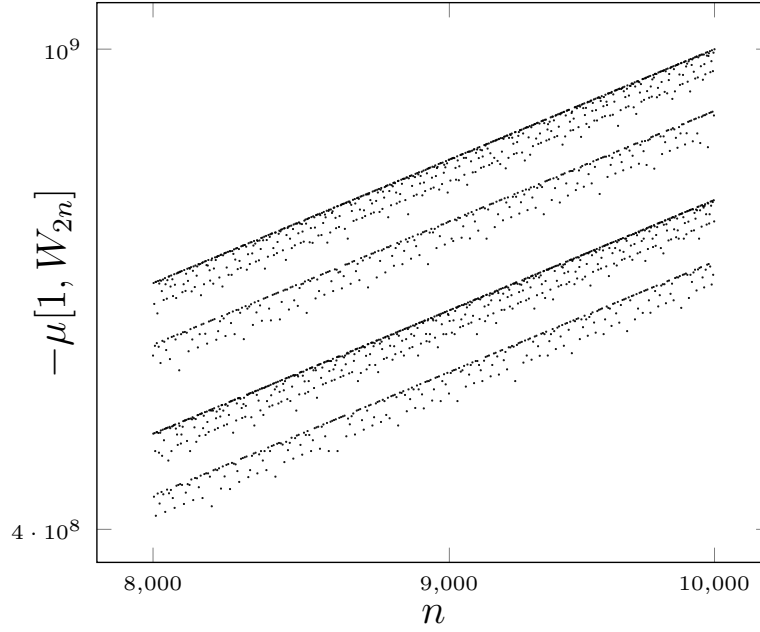
Figure 4.6 is a log-log plot of the values of  $-\mu[1, W_{2n}]$  from  $n = 8,000$  to  $n = 10,000$ . As can be seen, there seems to be some evidence that the values fall into distinct bands, and we have confirmed that this pattern continues up to  $n = 1,000,000$ . Examination of the values of  $\mu[1, W_{2n-1}]$  reveals the same patterns.

Following discussions at Permutation Patterns 2017, Vít Jelínek [24] provided the following conjecture (rephrased to reflect our notation).

**Conjecture 24** (Jelínek [24]). Let  $M(n)$  denote the absolute value of the Möbius function  $\mu[1, W_n] = \mu[1, M_n]$ . Then for  $n > 50$  we have

$$\begin{aligned} M(2n) &= n^2 \iff n+1 \text{ is prime and } n \equiv 0 \pmod{6} \\ M(2n) &= n^2 - 1 \iff n+1 \text{ is prime and } n \equiv 4 \pmod{6} \\ M(2n+1) &= n^2 - n \iff n+1 \text{ is prime and } n \equiv 0 \pmod{6} \\ M(2n+1) &= n^2 - n - 1 \iff n+1 \text{ is prime and } n \equiv 4 \pmod{6} \end{aligned}$$

Further, Jelínek notes that there does not seem to be any other small constant  $k$

Figure 4.6: Log-Log plot of  $|W_{2n}|$ .

such that  $M(n) = (n^2 - k)/4$  infinitely often.

We also have the following conjecture relating to the banding of the values.

**Conjecture 25.** Let  $M(n)$  denote the absolute value of the Möbius function  $\mu[1, W_n] = \mu[1, M_n]$ . Let  $E(n) = M(n)/(n^2)$ , and let  $O(n) = M(n)/(n^2 + n)$ . Then, with  $n \geq 1$ , there exist constants  $0 < a < b < c < d < e < f < g < 1$  such that

$$\begin{array}{ll}
 E(12n + 10) \in [a, b] & O(12n + 11) \in [a, b] \\
 E(12n + 2) \in [c, d] & O(12n + 3) \in [c, d] \\
 E(12n + 6) \in [c, d] & O(12n + 7) \in [c, d] \\
 E(12n + 4) \in [e, f] & O(12n + 5) \in [e, f] \\
 E(12n + 8) \in [g, 1] & O(12n + 9) \in [g, 1] \\
 E(12n) \in [g, 1] & O(12n + 1) \in [g, 1]
 \end{array}$$

Examining the first 2,000,000 values of  $\mu[1, W_n]$  gives the following estimates for the constants.

$a$	$b$	$c$	$d$	$e$	$f$	$g$
0.615	0.680	0.692	0.760	0.821	0.896	0.923

The *complete nearly-layered* permutations are formed by interleaving descending permutations. Formally, a complete nearly-layered permutation has the form

$$\alpha_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_{k-1} \otimes \alpha_k$$

where each  $\alpha_i$  is a descending permutation, with  $\alpha_i > 1$  for  $i = 2, \dots, k-1$ . If we set  $\alpha_i = 21$  for  $i = 2, \dots, k-1$ , and  $\alpha_1, \alpha_k \in \{1, 21\}$ , then we obtain the increasing oscillations.

The computational approach taken for increasing oscillations could, we think, be adapted to complete nearly-layered permutations. It is clear that the equivalent of the inequalities in Lemmas 20 and 22 would be somewhat more complex than those found here, but we believe that it should be possible to define an algorithm that could determine the Möbius function for complete nearly-layered permutations where the lower bound is sum indecomposable.

Average number of . . .	Value
Permutations	3373
Sum-indecomposable permutations	2492
Skew-indecomposable permutations	2445

Table 4.2: Statistics for 1000 posets  $[1, \pi)$  defined by a random permutation of length 14, with figures rounded to the nearest integer.

## 4.7 Chapter summary

We started this chapter by saying that our motivation was to find a contributing set  $\mathfrak{C}_{\sigma, \pi}$  that is significantly smaller than the poset interval  $[\sigma, \pi)$ , and a  $\{0, \pm 1\}$  weighting function  $W(\sigma, \alpha, \pi)$  such that

$$\mu[\sigma, \pi] = - \sum_{\alpha \in \mathfrak{C}_{\sigma, \pi}} \mu[\sigma, \alpha] W(\sigma, \alpha, \pi).$$

Our results are, essentially, computational. By this we mean that if  $\sigma$  is indecomposable, then Theorem 14 can be used to compute the value of the Möbius function on an interval  $[\sigma, \pi]$  with less computational resources than that required by the standard recursion of Equation 2.1. During the preparation of this thesis the author generated 1000 random permutations with length 14, then determined the poset  $[1, \pi)$  defined by the random permutation  $(\pi)$ , and then counted the permutations in the poset, the sum-indecomposable permutations in the poset, and the skew-indecomposable permutations in the poset. The details are summarised in Table 4.2.

These statistics seem to indicate that the improvement, at least for small intervals, does not seem to be as significant as the author hoped. With hindsight this is not unexpected. The proportion of permutations that are sum or skew decomposable tends to zero as the length of the permutation increases, from which one can readily deduce that the proportion of strongly indecomposable permutations must tend to 1 as the length of the permutation increases. This essentially means that the size of the contributing set  $\mathfrak{C}_{\sigma, \pi}$  is likely to be only slightly smaller than the size of the overall poset.

The author wrote a computer program, Permutation WorkShop (PWS) [32], which can be used to investigate the Möbius function on the permutation poset. The author found that the overhead of identifying which permutations were in the contributing set  $\mathfrak{C}_{\sigma,\pi}$ , and the overhead of calculating  $W(\sigma, \alpha, \pi)$  meant that, in general, calculations for permutations with an indecomposable lower bound took longer than using the standard recursive definition in Equation 2.1. This observation is, however, limited to the way in which PWS operates, and, indeed, the hardware it runs on. We suspect, however, that this observation is likely to be applicable to other routines that calculate the Möbius function on the permutation poset.

Despite the comments above, we still feel that the overall approach of finding a contributing set and a weighting function so that we can write

$$\mu[\sigma, \pi] = - \sum_{\alpha \in \mathfrak{C}_{\sigma,\pi}} \mu[\sigma, \alpha] W(\sigma, \alpha, \pi).$$

is valid, and indeed, although it is not phrased in these terms, the results in Chapter 6 use this method successfully.

By contrast, when we consider intervals  $[\sigma, \pi]$  where  $\pi$  is an increasing oscillation, we find that the number of indecomposable permutations contained in  $W(n)$  or  $M(n)$  is, apart from trivial values of  $\pi$ , no greater than  $2n - 4$ , and this upper bound is only achieved when  $\sigma = 1$ . This is because any indecomposable permutation contained in an increasing oscillation must itself be a (smaller) increasing oscillation, and there are only two increasing oscillations of each length. It is well-known that, for most intervals  $[1, \pi]$ , the number of permutations in the poset grows exponentially as  $|\pi|$  increases, and so we believe that in this specific case we have indeed found a contributing set that is significantly smaller than the poset. As supporting evidence for this claim, we note that we were easily able to calculate  $\mu[\pi]$  where  $\pi$  was an increasing oscillation with 2,000,000 elements.

This gave us the raw data to notice the banding shown in Figure 4.6, and led to Conjecture 25. We are not aware of any other set of permutations with a simple length-based construction where the values of the Möbius function fall into bands as the length of the permutation(s) increase. The Möbius function on the permutation

pattern poset is, however, notoriously hard to compute in general, so it is quite possible that such sets do exist, but we do not have the understanding and/or the technology to be able to calculate values that would exhibit banding.

We are not the only researchers to have considered the behaviour of  $\mu[W_n]$ , and Conjecture 24 comes from a personal communication with Vít Jelínek [24]. We have used our computations of  $\mu[W_n]$  to confirm that this conjecture holds for  $50 < n \leq 2,000,000$ .

We suspect that the banding behaviour in Conjecture 25 is a consequence of the link with the prime numbers in Conjecture 24. While there are results that give us expressions for the principal Möbius function value, we are not aware of any result, whether relating to the value of the principal Möbius function, or the growth of the principal Möbius function, where the result has a link to the prime numbers. This suggest to us that one possible area for future research would be to develop a better understanding of the behaviour of the principal Möbius function of increasing oscillations. We would hope that if we could find a relationship that accounted for the apparent link with prime numbers, then we would also have a better understanding of the permutation pattern poset.

# Chapter 5

## Zeros of the Möbius function of permutations

### 5.1 Preamble

This chapter is based on a published paper [\[15\]](#), which is joint work with Robert Brignall, Vít Jelínek and Jan Kynčl.

In this chapter we show that if a permutation  $\pi$  contains two intervals of length 2, where one interval is an ascent and the other a descent, then the Möbius function  $\mu[1, \pi]$  of the interval  $[1, \pi]$  is zero. As a consequence, we prove that the proportion of permutations of length  $n$  with principal Möbius function equal to zero is asymptotically bounded below by  $(1 - 1/e)^2 \geq 0.3995$ . This is the first result determining the value of  $\mu[1, \pi]$  for an asymptotically positive proportion of permutations  $\pi$ .

We further establish other general conditions on a permutation  $\pi$  that ensure  $\mu[1, \pi] = 0$  including the occurrence in  $\pi$  of any interval of the form  $\alpha \oplus 1 \oplus \beta$ .



## 5.2 Introduction

In this section we describe our principal results, and give an overview of the previous work in this area. Formal definitions are given in the next section.

In this chapter, we are mainly concerned with the principal Möbius function. We focus on the zeros of the principal Möbius function, that is, on the permutations  $\pi$  for which  $\mu[\pi] = 0$ . We show that we can often determine that a permutation  $\pi$  is such a Möbius zero by examining small localities of  $\pi$ . We formalize this idea using the notion of an “annihilator”. Informally, an annihilator is a permutation  $\alpha$  such that any permutation  $\pi$  containing an interval copy of  $\alpha$  is a Möbius zero. We will describe an infinite family of annihilators.

We will also prove that any permutation containing an increasing as well as a decreasing interval of size 2 is a Möbius zero. Based on this result, we show that the asymptotic proportion of Möbius zeros among the permutations of a given length is at least  $(1 - 1/e)^2 \geq 0.3995$ . This is the first known result determining the values of the principal Möbius function for an asymptotically positive fraction of permutations. We will also demonstrate how our results on the principal Möbius function can be extended to intervals whose lower bound is not 1.

Burstein, Jelínek, Jelínková and Steingrímsson [18] found a recursion for the Möbius function for sum and skew decomposable permutations. They used this to determine the Möbius function for separable permutations. Their results for sum and skew decomposable permutations implicitly include a result that only concerns small localities, which is that, up to symmetry, if a permutation  $\pi$  of length greater than two begins 12, then  $\mu[\pi] = 0$ .

Smith [44] found an explicit formula for the Möbius function on the interval  $[1, \pi]$  for all permutations  $\pi$  with a single descent. Smith’s paper includes a lemma stating that if a permutation  $\pi$  contains an interval order-isomorphic to 123, then  $\mu[\pi] = 0$ . While the result in [18] requires that the permutation starts with a particular sequence, Smith’s result is, in some sense, more general, as the critical interval (123) can occur in any position. Smith’s lemma may be viewed as the first instance of an

---

annihilator result. Our results on annihilators provide a common generalization of Smith's lemma and the above mentioned result of Burstein et al. [\[18\]](#).

### 5.3 Definitions and notation

Recall that an *adjacency* in a permutation is an interval of length two. If a permutation contains a monotonic interval of length three or more, then each subinterval of length two is an adjacency. As examples, 367249815 has two adjacencies, 67 and 98; and 1432 also has two adjacencies, 43 and 32. If an adjacency is ascending, then it is an *up-adjacency*, otherwise it is a *down-adjacency*.

If a permutation  $\pi$  contains at least one up-adjacency, and at least one down-adjacency, then we say that  $\pi$  has *opposing adjacencies*. An example of a permutation with opposing adjacencies is 367249815, which is shown in Figure 5.1.

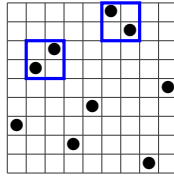


Figure 5.1: A permutation with opposing adjacencies.

A permutation that does not contain any adjacencies is *adjacency-free*. Some early papers use the term “strongly irreducible” for what we call adjacency-free permutations. See, for example, Atkinson and Stitt [3].

Given a permutation  $\sigma$  of length  $n$ , and permutations  $\alpha_1, \dots, \alpha_n$ , not all of them equal to the empty permutation  $\epsilon$ , the *inflation* of  $\sigma$  by  $\alpha_1, \dots, \alpha_n$ , written as  $\sigma[\alpha_1, \dots, \alpha_n]$ , is the permutation obtained by removing the element  $\sigma_i$  if  $\alpha_i = \epsilon$ , and replacing  $\sigma_i$  with an interval isomorphic to  $\alpha_i$  otherwise. Note that this is slightly different to the standard definition of inflation, originally given in Albert and Atkinson [1], which does not allow inflation by the empty permutation. As examples,  $3624715[1, 12, 1, 1, 21, 1, 1] = 367249815$ , and  $3624715[\epsilon, 1, 1, \epsilon, 1, \epsilon, 1] = 3142$ .

In many cases we will be interested in permutations where most positions are inflated by the singleton permutation 1. If  $\sigma = 3624715$ , then we will write  $\sigma[1, 12, 1, 1, 21, 1, 1] = 367249815$  as  $\sigma_{2,5}[12, 21]$ . Formally,  $\sigma_{i_1, \dots, i_k}[\alpha_1, \dots, \alpha_k]$  is the

inflation of  $\sigma$  where  $\sigma_{i_j}$  is inflated by  $\alpha_j$  for  $j = 1, \dots, k$ , and all other positions of  $\sigma$  are inflated by 1. When using this notation, we always assume that the indices  $i_1, \dots, i_k$  are distinct; however, we make no assumption about their relative order.

Our aim is to study the Möbius function of the permutation poset, that is, the poset of finite permutations ordered by containment. We are interested in describing general examples of intervals  $[\sigma, \pi]$  such that  $\mu[\sigma, \pi] = 0$ , with particular emphasis on the case  $\sigma = 1$ . We say that  $\pi$  is a *Möbius zero* (or just *zero*) if  $\mu[\pi] = 0$ , and we say that  $\pi$  is a  $\sigma$ -*zero* if  $\mu[\sigma, \pi] = 0$ .

It turns out that many sufficient conditions for  $\pi$  to be a Möbius zero can be stated in terms of inflations. We say that a permutation  $\phi$  is an *annihilator* if every permutation that has an interval copy of  $\phi$  is a Möbius zero; in other words, for every  $\tau$  and every  $i \leq |\tau|$  the permutation  $\tau_i[\phi]$  is a Möbius zero. More generally, we say that  $\phi$  is a  $\sigma$ -*annihilator* if every permutation with an interval copy of  $\phi$  is a  $\sigma$ -zero.

We say that a pair of permutations  $\phi, \psi$  is an *annihilator pair* if for every permutation  $\tau$  and every pair of distinct indices  $i, j \leq |\tau|$ , the permutation  $\tau_{i,j}[\phi, \psi]$  is a Möbius zero.

Observe that for an annihilator  $\phi$ , any permutation containing an interval copy of  $\phi$  is also an annihilator. Likewise, if  $\phi$  and  $\psi$  form an annihilator pair then any permutation containing disjoint interval copies of  $\phi$  and  $\psi$  is an annihilator.

As our first main result, presented in Section 5.4, we show that the two permutations 12 and 21 are an annihilator pair, or equivalently, any permutation with opposing adjacencies is a Möbius zero. Later, in Section 5.6, we use this result to prove that Möbius zeros have asymptotic density at least  $(1 - 1/e)^2$ .

We also prove that for any two non-empty permutations  $\alpha$  and  $\beta$ , the permutation  $\alpha \oplus 1 \oplus \beta = 123[\alpha, 1, \beta]$  is an annihilator, and generalize this result to a construction of  $\sigma$ -annihilators for general  $\sigma$ . These results are presented in Section 5.5.

Finally, in Section 5.7, we give several examples of annihilators and annihilator pairs that do not directly follow from the results in the previous sections.

### 5.3.1 Intervals with vanishing Möbius function

We will now present several basic facts about the Möbius function, which are valid in an arbitrary finite poset. The first fact is a simple observation following directly from the definition of the Möbius function, and we present it without proof.

**Fact 26.** *Let  $P$  be a finite poset with Möbius function  $\mu_P$ , and let  $x$  and  $y$  be two elements of  $P$  satisfying  $\mu_P[x, y] = 0$ . Let  $Q$  be the poset obtained from  $P$  by deleting the element  $y$ , and let  $\mu_Q$  be its Möbius function. Then for every  $z \in Q$ , we have  $\mu_Q[x, z] = \mu_P[x, z]$ .*

Next, we introduce two types of intervals whose specific structure ensures that their Möbius function is zero.

Let  $[x, y]$  be a finite interval in a poset  $P$ . We say that  $[x, y]$  is *narrow-tipped* if it contains an element  $z$  different from  $x$  such that  $[x, y] = [x, z]$ . The element  $z$  is then called the *core* of  $[x, y]$ .

We say that the interval  $[x, y]$  is *diamond-tipped* if there are three elements  $z, z'$  and  $w$ , all different from  $x$ , and such that

1.  $[x, y] = [x, z] \cup [x, z']$  and
2.  $[x, z] \cap [x, z'] = [x, w]$ .

Condition 2 is equivalent to  $w$  being the greatest lower bound of  $z$  and  $z'$  in the interval  $[x, y]$ . The triple of elements  $(z, z', w)$  is again called the *core* of  $[x, y]$ . Figure 5.2 shows examples of narrow-tipped and diamond-tipped posets.

**Fact 27.** *Let  $P$  be a poset with Möbius function  $\mu_P$ , and let  $[x, y]$  be a finite interval in  $P$ . If  $[x, y]$  is narrow-tipped or diamond-tipped, then  $\mu_P[x, y] = 0$ .*

*Proof.* If  $[x, y]$  is narrow-tipped with core  $z$ , then

$$\mu_P[x, y] = - \sum_{v \in [x, y)} \mu_P[x, v] = - \sum_{v \in [x, z)} \mu_P[x, v] = 0.$$

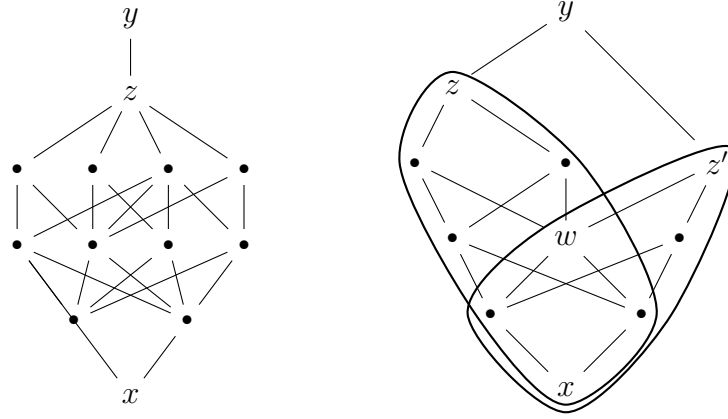


Figure 5.2: Examples of narrow-tipped (left) and diamond-tipped (right) posets.

If  $[x, y]$  is diamond-tipped with core  $(z, z', w)$  then

$$\begin{aligned}
 \mu_P[x, y] &= - \sum_{v \in [x, y)} \mu_P[x, v] \\
 &= - \sum_{v \in [x, z] \cup [x, z']} \mu_P[x, v] \\
 &= - \sum_{v \in [x, z]} \mu_P[x, v] - \sum_{v \in [x, z']} \mu_P[x, v] + \sum_{v \in [x, z] \cap [x, z']} \mu_P[x, v] \\
 &= - \sum_{v \in [x, z]} \mu_P[x, v] - \sum_{v \in [x, z']} \mu_P[x, v] + \sum_{v \in [x, w]} \mu_P[x, v] \\
 &= 0.
 \end{aligned}$$

□

### 5.3.2 Embeddings

Recall that an *embedding* of a permutation  $\sigma \in \mathcal{S}_k$  into a permutation  $\pi \in \mathcal{S}_n$  is a function  $f: [k] \rightarrow [n]$  with the following properties:

- $1 \leq f(1) < f(2) < \dots < f(k) \leq n$ .
- For any  $i, j \in [k]$ , we have  $\sigma_i < \sigma_j$  if and only if  $\pi_{f(i)} < \pi_{f(j)}$ .

We let  $\mathcal{E}(\sigma, \pi)$  denote the set of embeddings of  $\sigma$  into  $\pi$ , and  $E(\sigma, \pi)$  denote the cardinality of  $\mathcal{E}(\sigma, \pi)$ .

For an embedding  $f$  of  $\sigma$  into  $\pi$ , the *image* of  $f$ , denoted  $\text{Img}(f)$ , is the set  $\{f(i); i \in [k]\}$ . In particular,  $|\text{Img}(f)| = |\sigma|$ . The permutation  $\sigma$  is the *source* of the embedding  $f$ , denoted  $\text{src}_\pi(f)$ . When  $\pi$  is clear from the context (as it usually will be) we write  $\text{src}(f)$  instead of  $\text{src}_\pi(f)$ . Note that for a fixed  $\pi$ , the set  $\text{Img}(f)$  determines both  $f$  and  $\text{src}_\pi(f)$  uniquely.

We say that an embedding  $f$  is *even* if the cardinality of  $\text{Img}(f)$  is even, otherwise  $f$  is *odd*. In our arguments, we will frequently consider *sign-reversing* mappings on sets of embeddings (with different sources), which are mappings that map an odd embedding to an even one and vice versa. A typical example of a sign-reversing mapping is the so-called *i-switch*, which we now define. For a permutation  $\pi \in \mathcal{S}_n$ , let  $\mathcal{E}(*, \pi)$  be the set  $\bigcup_{\sigma \leq \pi} \mathcal{E}(\sigma, \pi)$ . For an index  $i \in [n]$ , the *i-switch* of an embedding  $f \in \mathcal{E}(*, \pi)$ , denoted  $\Delta_i(f)$ , is the embedding  $g \in \mathcal{E}(*, \pi)$  uniquely determined by the following properties:

$$\begin{aligned} \text{Img}(g) &= \text{Img}(f) \cup \{i\} \text{ if } i \notin \text{Img}(f), \text{ and} \\ \text{Img}(g) &= \text{Img}(f) \setminus \{i\} \text{ if } i \in \text{Img}(f). \end{aligned}$$

For example, consider the permutations  $\sigma = 132$  and  $\pi = 41253$ , and the embedding  $f \in \mathcal{E}(\sigma, \pi)$  satisfying  $f(1) = 2$ ,  $f(2) = 4$ , and  $f(3) = 5$ . We then have  $\text{Img}(f) = \{2, 4, 5\}$ . Defining  $g = \Delta_3(f)$ , we see that  $\text{Img}(g) = \{2, 3, 4, 5\}$ , and  $\text{src}(g)$  is the permutation 1243. Similarly, for  $h = \Delta_5(g)$ , we have  $\text{Img}(h) = \{2, 3, 4\}$  and  $\text{src}(h) = 123$ .

Note that for any  $\pi \in \mathcal{S}_n$  and any  $i \in [n]$ , the function  $\Delta_i$  is a sign-reversing involution on the set  $\mathcal{E}(*, \pi)$ .

Consider, for a given  $\pi \in \mathcal{S}_n$ , two embeddings  $f, g \in \mathcal{E}(*, \pi)$ . We say that  $f$  is *contained in*  $g$  if  $\text{Img}(f) \subseteq \text{Img}(g)$ . Note that if  $f$  is contained in  $g$ , then the permutation  $\text{src}(f)$  is contained in  $\text{src}(g)$ , and if a permutation  $\lambda$  is contained in a permutation  $\tau$ , then any embedding from  $\mathcal{E}(\tau, \pi)$  contains at least one embedding from  $\mathcal{E}(\lambda, \pi)$ . In particular, the mapping  $f \mapsto \text{src}(f)$  is a poset homomorphism from

the set  $\mathcal{E}(*, \pi)$  ordered by containment onto the interval  $[\epsilon, \pi]$  in the permutation pattern poset.

### 5.3.3 Möbius function via normal embeddings

We will now derive a general formula which will become useful in several subsequent arguments. The formula can be seen as a direct consequence of the well-known Möbius inversion formula. The following form of the Möbius inversion formula can be deduced, for example, from Proposition 3.7.2 in Stanley [50]. Recall that a poset is *locally finite* if each of its intervals is finite.

**Fact 28** (Möbius inversion formula). *Let  $P$  be a locally finite poset with maximum element  $y$ , let  $\mu$  be the Möbius function of  $P$ , and let  $F: P \rightarrow \mathbb{R}$  be a function. If a function  $G: P \rightarrow \mathbb{R}$  is defined by*

$$G(x) = \sum_{z \in [x, y]} F(z),$$

*then for every  $x \in P$ , we have*

$$F(x) = \sum_{z \in [x, y]} \mu[x, z] G(z).$$

As a consequence, we obtain the following result.

**Proposition 29.** *Let  $\sigma$  and  $\pi$  be arbitrary permutations, and let  $F: [\sigma, \pi] \rightarrow \mathbb{R}$  be a function satisfying  $F(\pi) = 1$ . We then have*

$$\mu[\sigma, \pi] = F(\sigma) - \sum_{\lambda \in [\sigma, \pi]} \mu[\sigma, \lambda] \sum_{\tau \in [\lambda, \pi]} F(\tau). \quad (5.1)$$

*Proof.* Fix  $\sigma, \pi$  and  $F$ . For  $\lambda \in [\sigma, \pi]$ , define  $G(\lambda) = \sum_{\tau \in [\lambda, \pi]} F(\tau)$ . Using Fact 28 for the poset  $P = [\sigma, \pi]$ , we obtain

$$F(\sigma) = \sum_{\lambda \in [\sigma, \pi]} \mu[\sigma, \lambda] G(\lambda).$$



Substituting the definition of  $G(\lambda)$  into the above identity and noting that  $F(\pi) = 1$ , we get

$$\begin{aligned} F(\sigma) &= \sum_{\lambda \in [\sigma, \pi]} \mu[\sigma, \lambda] \sum_{\tau \in [\lambda, \pi]} F(\tau) \\ &= \mu[\sigma, \pi] + \sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda] \sum_{\tau \in [\lambda, \pi]} F(\tau), \end{aligned}$$

from which the proposition follows.  $\square$

In our applications, the function  $F(\tau)$  will usually be defined in terms of the number of embeddings of  $\tau$  into  $\pi$  satisfying certain additional conditions. We call embeddings that satisfy these conditions *normal embeddings*. We briefly discussed normal embeddings in Chapter 2. We extend that discussion here, as, in this thesis, normal embeddings are only used in the current chapter.

The notion of normal embedding seems to originate from the work of Björner [7], who defined normal embeddings between words, and showed that in the subword order of words over a finite alphabet, the Möbius function of any interval  $[x, y]$  is equal in absolute value to the number of normal embeddings of  $x$  into  $y$ .

Björner's approach was later extended to the computation of the Möbius function in the composition poset [39], the poset of separable permutations [18], or the poset of permutations with a fixed number of descents [45]. In all these cases, the authors define a notion of “normal” embeddings tailored for their poset, and then express the Möbius function of an interval  $[x, y]$  as the sum of weights of the “normal” embeddings of  $x$  into  $y$ , where each normal embedding has weight 1 or  $-1$ .

For general permutations, this simple approach fails, since the Möbius function  $\mu[\sigma, \pi]$  is sometimes larger than the number of all embeddings of  $\sigma$  into  $\pi$ . However, Smith [46] introduced a notion of normal embedding applicable to arbitrary permutations, and proved a formula expressing  $\mu[\sigma, \pi]$  as a summation over certain sets of normal embeddings.

For consistency, we adopt the term “normal embedding” in this chapter, although in our proofs, we will need to introduce several notions of normality, which are different

from each other and from the notions of normality introduced by previous authors. We will always use  $\mathcal{NE}(\tau, \pi)$  to denote the set of embeddings of  $\tau$  into  $\pi$  satisfying the definition of normality used in the given context, and we let  $|\mathcal{NE}(\tau, \pi)|$  be the cardinality of  $\mathcal{NE}(\tau, \pi)$ .

The next proposition provides a general basis for all our subsequent applications of normal embeddings.

**Proposition 30.** *Let  $\sigma$  and  $\pi$  be permutations. Suppose that for each  $\tau \in [\sigma, \pi]$  we fix a subset  $\mathcal{NE}(\tau, \pi)$  of  $\mathcal{E}(\tau, \pi)$ , with the elements of  $\mathcal{NE}(\tau, \pi)$  being referred to as normal embeddings of  $\tau$  into  $\pi$ . Assume that  $\mathcal{NE}(\pi, \pi) = \mathcal{E}(\pi, \pi)$ , that is, the unique embedding of  $\pi$  into  $\pi$  is normal. For each  $\lambda \in [\sigma, \pi)$ , define the two sets of embeddings*

$$\begin{aligned}\mathcal{NE}_\lambda(\text{odd}, \pi) &= \bigcup_{\substack{\tau \in [\lambda, \pi] \\ |\tau| \text{ odd}}} \mathcal{NE}(\tau, \pi) \quad \text{and} \\ \mathcal{NE}_\lambda(\text{even}, \pi) &= \bigcup_{\substack{\tau \in [\lambda, \pi] \\ |\tau| \text{ even}}} \mathcal{NE}(\tau, \pi).\end{aligned}$$

If for every  $\lambda \in [\sigma, \pi)$  such that  $\mu[\sigma, \lambda] \neq 0$ , we have the identity

$$|\mathcal{NE}_\lambda(\text{odd}, \pi)| = |\mathcal{NE}_\lambda(\text{even}, \pi)|, \quad (5.2)$$

then  $\mu[\sigma, \pi] = (-1)^{|\pi| - |\sigma|} |\mathcal{NE}(\sigma, \pi)|$ .

*Proof.* The trick is to define the function  $F(\tau) = (-1)^{|\pi| - |\tau|} |\mathcal{NE}(\tau, \pi)|$  and apply Proposition 29. This yields

$$\begin{aligned}\mu[\sigma, \pi] &= F(\sigma) - \sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda] \sum_{\tau \in [\lambda, \pi]} F(\tau) \\ &= F(\sigma) - \sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda] \sum_{\tau \in [\lambda, \pi]} (-1)^{|\pi| - |\tau|} |\mathcal{NE}(\tau, \pi)| \\ &= F(\sigma) - \sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda] (-1)^{|\pi|} (|\mathcal{NE}_\lambda(\text{even}, \pi)| - |\mathcal{NE}_\lambda(\text{odd}, \pi)|) \\ &= F(\sigma)\end{aligned}$$

$$= (-1)^{|\pi| - |\sigma|} \text{NE}(\sigma, \pi),$$

as claimed. □

We remark that the general formula of Proposition 29 can be useful even in situations where the more restrictive assumptions of Proposition 30 fail. An example of such an application of Proposition 29 appears in Jelínek, Kantor, Kynčl and Tancer [25].

## 5.4 Permutations with opposing adjacencies

In this section, we show that if a permutation has opposing adjacencies, then the value of the principal Möbius function is zero.

**Theorem 31.** *If  $\pi$  has opposing adjacencies, then  $\mu[\pi] = 0$ .*

For this theorem, we are able to give two proofs. One of them is based on the notion of diamond-tipped intervals, and the other uses the approach of normal embeddings. As both these approaches will later be adapted to more complicated settings, we find it instructive to include both proofs here.

*Proof via diamond-tipped posets.* For contradiction, suppose that the theorem fails, and let  $\pi$  be a shortest permutation with opposing adjacencies such that  $\mu[\pi] \neq 0$ . Since  $\pi$  has opposing adjacencies, there is a permutation  $\tau$  and indices  $i, j \leq |\tau|$  such that  $\pi = \tau_{i,j}[12, 21]$ . Define  $\phi = \tau_{i,j}[1, 21]$  and  $\phi' = \tau_{i,j}[12, 1]$ .

We claim that the interval  $[1, \pi]$  can be transformed into a diamond-tipped interval with core  $(\phi, \phi', \tau)$  by deleting a set of Möbius zeros from the interior of  $[1, \pi]$ . Since by Fact 26, the deletion of Möbius zeros does not affect the value of  $\mu[1, \pi]$ , and since diamond-tipped intervals have zero Möbius function by Fact 27, this claim will imply that  $\mu[1, \pi] = 0$ , a contradiction.

To prove the claim, note first that any permutation  $\lambda \in [1, \pi)$  with opposing adjacencies is a Möbius zero, since  $\pi$  is a minimal counterexample to the theorem. Choose any  $\lambda \in [1, \pi)$ . Observe that if  $\lambda$  has no up-adjacency, then  $\lambda \leq \phi$ , and symmetrically, if  $\lambda$  has no down-adjacency, then  $\lambda \leq \phi'$ . Thus, any  $\lambda \in [1, \pi)$  not belonging to  $[1, \phi] \cup [1, \phi']$  has opposing adjacencies and can be deleted from  $[1, \pi]$ .

Next, suppose that a permutation  $\lambda$  is in  $[1, \phi] \cap [1, \phi']$  but not in  $[1, \tau]$ . Observe that any permutation in  $[1, \phi] \setminus [1, \tau]$  has a down-adjacency, while any permutation in  $[1, \phi'] \setminus [1, \tau]$  has an up-adjacency. It follows that  $\lambda$  has opposing adjacencies and can again be deleted from  $[1, \pi]$ .

After these deletions, the remaining poset is diamond-tipped with core  $(\phi, \phi', \tau)$  as claimed, hence  $\mu[1, \pi] = 0$ , a contradiction.  $\square$

*Proof via normal embeddings.* Suppose again that  $\pi \in \mathcal{S}_n$  is a shortest counterexample. Suppose that  $\pi$  has an up-adjacency at positions  $i, i + 1$ , and a down-adjacency at positions  $j, j + 1$ . Note that the positions  $i, i + 1, j$  and  $j + 1$  are all distinct, and in particular  $n \geq 4$ .

We will say that an embedding  $f \in \mathcal{E}(*, \pi)$  is *normal* if  $\text{Img}(f)$  is a superset of  $[n] \setminus \{i, j\}$ . In other words,  $\text{Img}(f)$  contains all positions of  $\pi$  with the possible exception of  $i$  and  $j$ . Thus, there are four normal embeddings.

We will use Proposition 30 with the above notion of normal embeddings and with  $\sigma = 1$ . Clearly, we have  $\mathcal{E}(\pi, \pi) = \mathcal{NE}(\pi, \pi)$ . The main task is to verify equation (5.2), that is, to show that for every  $\lambda \in [1, \pi)$  such that  $\mu[\lambda] \neq 0$  we have  $|\mathcal{NE}_\lambda(\text{odd}, \pi)| = |\mathcal{NE}_\lambda(\text{even}, \pi)|$ . To prove this identity, we let  $\mathcal{NE}_\lambda(*, \pi)$  denote the set  $\mathcal{NE}_\lambda(\text{odd}, \pi) \cup \mathcal{NE}_\lambda(\text{even}, \pi)$ , and we will provide a sign-reversing involution on  $\mathcal{NE}_\lambda(*, \pi)$ .

Choose a  $\lambda \in [1, \pi)$  with  $\mu[\lambda] \neq 0$ . It follows that  $\lambda$  does not have opposing adjacencies, otherwise it would be a counterexample shorter than  $\pi$ . Without loss of generality, assume that  $\lambda$  has no up-adjacency. We will prove that the  $i$ -switch operation  $\Delta_i$  is a sign-reversing involution on  $\mathcal{NE}_\lambda(*, \pi)$ .

It is clear that  $\Delta_i$  is sign-reversing. We need to demonstrate that for every  $f \in \mathcal{NE}_\lambda(*, \pi)$ , the embedding  $g = \Delta_i(f)$  is again in  $\mathcal{NE}_\lambda(*, \pi)$ . It is clear that  $g$  is normal. It remains to argue that  $\text{src}(g)$  contains  $\lambda$ , or in other words, that there is an embedding of  $\lambda$  into  $\pi$  contained in  $g$ . Let  $h$  be a (not necessarily normal) embedding of  $\lambda$  into  $\pi$  contained in  $f$ . If  $i$  is not in  $\text{Img}(h)$ , then  $h$  is also contained in  $g$ , and we are done. Suppose now that  $i \in \text{Img}(h)$ . Then  $i + 1 \notin \text{Img}(h)$ , because  $i$  and  $i + 1$  form an up-adjacency in  $\pi$  while  $\lambda$  has no up-adjacency. We modify the embedding  $h$  so that the element mapped to  $i$  will be mapped to  $i + 1$  instead, and the mapping of the remaining elements is unchanged; let  $h'$  be the resulting embedding (formally, we have  $\Delta_i(\Delta_{i+1}(h)) = h'$ ). Since  $i$  and  $i + 1$  form an adjacency in  $\pi$ , we have  $\text{src}(h') = \text{src}(h) = \lambda$ . Since  $i + 1$  is in the image of all normal embeddings, we see that  $h'$  is contained in  $g$ , and so  $g \in \mathcal{NE}_\lambda(*, \pi)$ . This shows that  $\Delta_i$  is the required sign-reversing involution on  $\mathcal{NE}_\lambda(*, \pi)$ , verifying the assumptions of Proposition 30.

Proposition 30 then gives us that  $\mu[1, \pi] = (-1)^{n-1} \text{NE}(1, \pi)$ . Since every normal

embedding into  $\pi$  contains both  $i + 1$  and  $j + 1$  in its image, there is clearly no normal embedding of 1 into  $\pi$  and therefore we get  $\mu[1, \pi] = 0$ .  $\square$

## 5.5 A general construction of $\sigma$ -annihilators

Let  $\sigma$  be a fixed non-empty lower bound permutation (the case  $\sigma = 1$  being the most interesting). Recall that a permutation  $\phi$  is a  $\sigma$ -zero if  $\mu[\sigma, \phi] = 0$ , and  $\phi$  is a  $\sigma$ -annihilator if every permutation with an interval copy of  $\phi$  is a  $\sigma$ -zero. Clearly, any  $\sigma$ -annihilator is also a  $\sigma$ -zero. Our goal in this section is to present a general construction of an infinite family of  $\sigma$ -annihilators.

A permutation  $\phi$  is  $\sigma$ -narrow if  $\phi$  contains a permutation  $\phi^-$  of size  $|\phi| - 1$  such that every permutation in the set  $[1, \phi) \setminus [1, \phi^-]$  is a  $\sigma$ -annihilator. In this situation, we call  $\phi^-$  a  $\sigma$ -core of  $\phi$ .

Note that if  $\phi$  is  $\sigma$ -narrow with  $\sigma$ -core  $\phi^-$ , then the interval  $[1, \phi]$  can be transformed into a narrow-tipped interval by a deletion of  $\sigma$ -annihilators. Our first goal is to show that, with a few exceptions, all  $\sigma$ -narrow permutations are  $\sigma$ -annihilators.

**Proposition 32.** *If a permutation  $\phi$  is  $\sigma$ -narrow with a  $\sigma$ -core  $\phi^-$ , and if  $\sigma$  has no interval copy of  $\phi$  or of  $\phi^-$ , then  $\phi$  is a  $\sigma$ -annihilator.*

*Proof.* Let  $\phi$  be  $\sigma$ -narrow with a  $\sigma$ -core  $\phi^-$ . Let  $\pi$  be a permutation with an interval copy of  $\phi$ , that is,  $\pi = \tau_i[\phi]$  for some  $\tau$  and  $i$ . We show that  $\mu[\sigma, \pi] = 0$ . We may assume that  $\sigma \leq \pi$ , otherwise  $\mu[\sigma, \pi] = 0$  trivially. Let  $\pi^-$  be the permutation  $\tau_i[\phi^-]$ . Note that  $\sigma \neq \pi$  and  $\sigma \neq \pi^-$ , since  $\sigma$  has no interval copy of  $\phi$  or of  $\phi^-$ .

The key step of the proof is to show that any permutation in  $[\sigma, \pi) \setminus [\sigma, \pi^-]$  is a  $\sigma$ -zero. After we have proved this, we may use Fact 26 to remove all such  $\sigma$ -zeros from the interval  $[\sigma, \pi]$  without affecting the value of  $\mu[\sigma, \pi]$ ; note that  $\sigma$  itself is clearly not a  $\sigma$ -zero, so it will not be removed, implying that  $\sigma < \pi^-$ . After the removal of  $[\sigma, \pi) \setminus [\sigma, \pi^-]$ , the remainder of the interval  $[\sigma, \pi]$  is a narrow-tipped poset with core  $\pi^-$ , yielding  $\mu[\sigma, \pi] = 0$  by Fact 27.

Therefore, to prove that  $\mu[\sigma, \pi] = 0$  for a particular  $\pi = \tau_i[\phi]$ , it is enough to show that all the permutations in  $[\sigma, \pi) \setminus [\sigma, \pi^-]$  are  $\sigma$ -zeros. We prove this by induction on  $|\tau|$ .

If  $|\tau| = 1$ , we have  $\pi = \phi$  and  $\pi^- = \phi^-$ . Then all the permutations in  $[1, \pi) \setminus [1, \pi^-]$  are  $\sigma$ -annihilators (and therefore  $\sigma$ -zeros) by definition of  $\sigma$ -narrowness, and in particular, restricting our attention to permutations containing  $\sigma$ , we see that all the permutations in  $[\sigma, \pi) \setminus [\sigma, \pi^-]$  are  $\sigma$ -zeros, as claimed.

Suppose that  $|\tau| > 1$ . Consider a permutation  $\gamma \in [\sigma, \pi) \setminus [\sigma, \pi^-]$ . Since  $\gamma$  is contained in  $\pi = \tau_i[\phi]$ , it can be expressed as  $\gamma = \tau_j^*[\phi^*]$  for some  $\epsilon \leq \phi^* \leq \phi$  and  $1 \leq \tau^* \leq \tau$ , where  $\tau^*$  has an embedding into  $\tau$  which maps  $j$  to  $i$ . Note that  $\phi^*$  cannot be contained in  $\phi^-$ , because in such case we would have  $\gamma \leq \pi^-$ . Moreover, if  $\phi^* = \phi$ , then necessarily  $\tau^* < \tau$ , and by induction  $\gamma$  is a  $\sigma$ -zero. Finally, if  $\phi^*$  is in  $[1, \phi) \setminus [1, \phi^-]$ , then  $\phi^*$  is a  $\sigma$ -annihilator by the  $\sigma$ -narrowness of  $\phi$ , and hence  $\gamma$  is a  $\sigma$ -zero.  $\square$

With the help of Proposition 32, we can now provide an explicit general construction of  $\sigma$ -annihilators.

**Proposition 33.** *Let  $\alpha$  and  $\beta$  be non-empty permutations. Assume that  $\sigma$  does not contain any interval copy of a permutation of the form  $\alpha' \oplus \beta'$  with  $1 \leq \alpha' \leq \alpha$  and  $1 \leq \beta' \leq \beta$  (in particular,  $\sigma$  has no up-adjacency). Then  $\alpha \oplus 1 \oplus \beta$  is  $\sigma$ -narrow with  $\sigma$ -core  $\alpha \oplus \beta$ , and  $\alpha \oplus 1 \oplus \beta$  is a  $\sigma$ -annihilator.*

*Proof.* We proceed by induction on  $|\alpha| + |\beta|$ . Suppose first that  $\alpha = \beta = 1$ . Then trivially  $\alpha \oplus 1 \oplus \beta = 123$  is  $\sigma$ -narrow with  $\sigma$ -core  $\alpha \oplus \beta = 12$ , since the set  $[1, 123) \setminus [1, 12]$  is empty. Moreover, by assumption,  $\sigma$  has no interval copy of 12, and therefore also no interval copy of 123, hence 123 is a  $\sigma$ -annihilator by Proposition 32.

Suppose now that  $|\alpha| + |\beta| > 2$ . Define  $\phi = \alpha \oplus 1 \oplus \beta$  and  $\phi^- = \alpha \oplus \beta$ . To prove that  $\phi$  is  $\sigma$ -narrow with  $\sigma$ -core  $\phi^-$ , we will show that any permutation  $\gamma \in [1, \phi) \setminus [1, \phi^-]$  is a  $\sigma$ -annihilator. Such a  $\gamma$  has the form  $\alpha' \oplus 1 \oplus \beta'$  for some  $1 \leq \alpha' \leq \alpha$  and  $1 \leq \beta' \leq \beta$ , with  $|\alpha'| + |\beta'| < |\alpha| + |\beta|$ ; note that we here exclude the cases  $\alpha' = \epsilon$  and  $\beta' = \epsilon$ , because in these cases  $\gamma$  would be contained in  $\phi^-$ . By induction,  $\gamma$  is  $\sigma$ -narrow, with  $\sigma$ -core  $\gamma^- = \alpha' \oplus \beta'$ . Moreover,  $\sigma$  has no interval isomorphic to  $\gamma$  or  $\gamma^-$ : observe that if  $\sigma$  had an interval isomorphic to  $\gamma$ , it would also have an interval isomorphic to  $\alpha' \oplus 1$ , which is forbidden by our assumptions on  $\sigma$ . Thus, we



may apply Proposition 32 to conclude that  $\gamma$  is a  $\sigma$ -annihilator, and in particular  $\phi$  is  $\sigma$ -narrow with  $\sigma$ -core  $\phi^-$ , as claimed. Proposition 32 then gives us that  $\phi$  is a  $\sigma$ -annihilator.  $\square$

Focusing on the special case  $\sigma = 1$ , which satisfies the assumptions of Proposition 33 trivially, we obtain the following result.

**Corollary 34.** *For any non-empty permutations  $\alpha$  and  $\beta$ , the permutation  $\alpha \oplus 1 \oplus \beta$  is an annihilator.*

## 5.6 The density of zeros

Our goal is to find an asymptotic positive lower bound on the proportion of permutations of length  $n$  whose principal Möbius function is zero. The key step is the following lemma.

**Lemma 35.** *Let  $s_n$  be the number of permutations of size  $n$  with opposing adjacencies. Then*

$$\frac{s_n}{n!} = \left(1 - \frac{1}{e}\right)^2 + O\left(\frac{1}{n}\right).$$

*Proof.* Let  $a_n$  be the number of permutations of size  $n$  that have no up-adjacency, and let  $b_n$  be the number of permutations of size  $n$  that have neither an up-adjacency nor a down-adjacency.

The numbers  $a_n$  (sequence A000255 in the OEIS [43]) have already been studied by Euler [23], and it is known [38] that they satisfy  $a_n/n! = e^{-1} + O(n^{-1})$ .

The numbers  $b_n$  (sequence A002464 in the OEIS [43]) satisfy the asymptotics  $b_n/n! = e^{-2} + O(n^{-1})$ , which follows from the results of Kaplansky [26] (see also Albert et al. [2]).

We may now express the number  $s_n$  of permutations with opposing adjacencies by inclusion-exclusion as follows: we subtract from  $n!$  the number of permutations having no up-adjacency and the number of permutations having no down-adjacency, and then we add back the number of permutations having no adjacency at all. This yields  $s_n = n! - 2a_n + b_n$ , from which the lemma follows by the above-mentioned asymptotics of  $a_n$  and  $b_n$ .  $\square$

Combining Theorem 31 with Lemma 35 we obtain the following consequence, which is the main result of this section.

**Corollary 36.** *For a given  $n$  and for  $\pi$  a uniformly random permutation of length  $n$ , the probability that  $\mu[\pi] = 0$  is at least*

$$\left(1 - \frac{1}{e}\right)^2 - O\left(\frac{1}{n}\right).$$

## 5.7 More complicated examples

We will now construct several specific examples of annihilators and annihilator pairs, which are not covered by the general results obtained in the previous sections. We begin with a construction of four new annihilator pairs, which we will later use to construct new annihilators.

**Theorem 37.** *The two permutations 213 and 2431 form an annihilator pair.*

*Proof.* Our proof is based on the concept of normal embeddings and follows a similar structure as the normal embedding proof of Theorem 31.

Suppose for contradiction that there is a permutation  $\pi$  that contains an interval isomorphic to 213 as well as an interval isomorphic to 2431, and that  $\mu[\pi] \neq 0$ . Fix a smallest possible  $\pi$ , and let  $n$  be its length. Note that an interval isomorphic to 213 is necessarily disjoint from an interval isomorphic to 2431, and in particular,  $n \geq 7$ .

Let  $i, i + 1$  and  $i + 2$  be three positions of  $\pi$  containing an interval copy of 213, and let  $j, j + 1, j + 2$  and  $j + 3$  be four positions containing an interval copy of 2431. We will apply the approach of Proposition 30, with  $\sigma = 1$ . We will say that an embedding  $f \in \mathcal{E}(*, \pi)$  is *normal* if  $\text{Img}(f)$  is a superset of  $[n] \setminus \{i + 2, j + 2, j + 3\}$ . Informally, the image of a normal embedding contains all the positions of  $\pi$ , except possibly some of the three positions that correspond to the value 3 of 213 or the values 3 and 1 of 2431 in the chosen interval copies of 213 and 2431, as shown in Figure 5.3. In particular, there are eight normal embeddings.

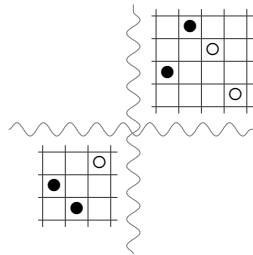


Figure 5.3: The intervals 213 and 2431 in Theorem 37. Normal embeddings may omit some of the hollow points.

We now verify the assumptions of Proposition 30. We obviously have  $\mathcal{NE}(\pi, \pi) = \mathcal{E}(\pi, \pi)$ . The main task is to verify, for a given  $\lambda \in [1, \pi)$  with  $\mu[\lambda] \neq 0$ , the identity (5.2) of Proposition 30, that is, the identity  $|\mathcal{NE}_\lambda(\text{odd}, \pi)| = |\mathcal{NE}_\lambda(\text{even}, \pi)|$ .

Fix a  $\lambda \in [1, \pi)$  such that  $\mu[\lambda] \neq 0$ , and let  $\mathcal{NE}_\lambda(*, \pi)$  be the set  $\mathcal{NE}_\lambda(\text{odd}, \pi) \cup \mathcal{NE}_\lambda(\text{even}, \pi)$ . We will describe a sign-reversing involution  $\Phi_\lambda$  on  $\mathcal{NE}_\lambda(*, \pi)$ . The involution  $\Phi_\lambda$  will always be equal to a switch operation  $\Delta_k$ , where the choice of  $k$  will depend on  $\lambda$ .

Suppose first that  $\lambda$  does not contain any down-adjacency. We claim that  $\Delta_{j+2}$  is an involution on the set  $\mathcal{NE}_\lambda(*, \pi)$ . To see this, choose  $f \in \mathcal{NE}_\lambda(*, \pi)$  and define  $g = \Delta_{j+2}(f)$ . It is clear that  $g$  is a normal embedding.

To prove that  $g$  belongs to  $\mathcal{NE}_\lambda(*, \pi)$ , it remains to show that  $\text{src}(g)$  contains  $\lambda$ , or equivalently, that there is an embedding of  $\lambda$  into  $\pi$  that is contained in  $g$ . Let  $h$  be an embedding of  $\lambda$  into  $\pi$  which is contained in  $f$ . If  $j+2 \notin \text{Img}(h)$ , then  $h$  is also contained in  $g$  and we are done.

Suppose then that  $j+2 \in \text{Img}(h)$ . This means that  $j+1$  is not in  $\text{Img}(h)$ , because  $\pi$  has a down-adjacency at positions  $j+1$  and  $j+2$ , while  $\lambda$  has no down-adjacency. We now modify  $h$  in such a way that the element previously mapped to  $j+2$  will be mapped to  $j+1$ , while the mapping of the remaining elements remains unchanged. Let  $h'$  be the embedding obtained from  $h$  by this modification; formally, we have  $h' = \Delta_{j+1}(\Delta_{j+2}(h))$ . Since the two elements  $\pi_{j+1}$  and  $\pi_{j+2}$  form an adjacency, we have  $\text{src}(h') = \text{src}(h) = \lambda$ . Moreover,  $h'$  is contained in  $g$  (recall that  $g$  is normal, and therefore  $\text{Img}(g)$  contains  $j+1$ ). Consequently,  $g$  is in  $\mathcal{NE}_\lambda(*, \pi)$ , as claimed.

We now deal with the case when  $\lambda$  contains a down-adjacency. Since  $\mu[\lambda] \neq 0$ , it follows by Theorem 31 that  $\lambda$  has no up-adjacency. We distinguish two subcases, depending on whether  $\lambda$  contains an interval copy of 2431.

Suppose that  $\lambda$  contains an interval copy of 2431. We will prove that in this case,  $\Delta_{i+2}$  is a sign-reversing involution on  $\mathcal{NE}_\lambda(*, \pi)$ . We begin by observing that  $\lambda$  has no interval copy of 213, otherwise  $\lambda$  would be a counterexample to Theorem 37, contradicting the minimality of  $\pi$ . Fix again an embedding  $f \in \mathcal{NE}_\lambda(*, \pi)$ , and define  $g = \Delta_{i+2}(f)$ . As in the previous case,  $g$  is clearly normal, and we only need to

show that there is an embedding of  $\lambda$  into  $\pi$  contained in  $g$ . Let  $h$  be an embedding of  $\lambda$  into  $\pi$  contained in  $f$ . If  $i+2 \notin \text{Img}(h)$ , then  $h$  is contained in  $g$  and we are done, so suppose  $i+2 \in \text{Img}(h)$ . If at least one of the two positions  $i$  and  $i+1$  belongs to  $\text{Img}(h)$ , then  $\lambda$  contains an up-adjacency or an interval copy of 213, contradicting our assumptions. Therefore, we can modify  $h$  so that the element mapped to  $i+2$  is mapped to  $i$  instead, obtaining an embedding of  $\lambda$  contained in  $g$  and showing that  $g \in \mathcal{NE}_\lambda(*, \pi)$ .

Finally, suppose that  $\lambda$  has no interval copy of 2431. In this case, we prove that  $\Delta_{j+3}$  is the required involution on  $\mathcal{NE}_\lambda(*, \pi)$ . As in the previous cases, we fix  $f \in \mathcal{NE}_\lambda(*, \pi)$ , define  $g = \Delta_{j+3}(f)$ , and let  $h$  be an embedding of  $\lambda$  contained in  $f$ ; we again want to modify  $h$  into an embedding  $\lambda$  contained in  $g$ . Let  $\alpha$  be the subpermutation of  $\lambda$  formed by those positions that are mapped into the set  $J = \{j, j+1, j+2, j+3\}$  by  $h$ . Recall that the positions in  $J$  induce an interval copy of 2431 in  $\pi$ . In particular,  $\alpha \leq 2431$ , and  $\lambda$  has an interval copy of  $\alpha$ . We know that  $\alpha \neq 2431$ , since we assume that  $\lambda$  has no interval copy of 2431. Also,  $\alpha \neq 321$ , since 321 is an annihilator by Corollary 34, while  $\mu[\lambda] \neq 0$ . Finally,  $\alpha \neq 231$ , since  $\lambda$  has no up-adjacency. This implies that  $\alpha \leq 132$ , and we can modify  $h$  so that all the positions originally mapped into  $J$  will get mapped into  $J \setminus \{j+3\}$ , obtaining an embedding of  $\lambda$  into  $\pi$  contained in  $g$ .

Having thus verified the assumptions of Proposition 30, we can conclude that  $\mu[\pi] = (-1)^{|\pi|-1} \text{NE}(1, \pi) = 0$ , a contradiction.  $\square$

The following three results are proved using similar methods to those used in the proof of Theorem 37.

**Theorem 38.** *The permutations 2143 and 2431 form an annihilator pair.*

**Theorem 39.** *The permutations 312 and 23514 form an annihilator pair.*

**Theorem 40.** *The permutations 25134 and 23514 form an annihilator pair.*

We omit the proofs here, as they were not included in the published paper [15] on which this chapter is based. They can be found in <https://arxiv.org/abs/1810.05449v1> [14].

**Theorem 41.** *Each of the three permutations 215463, 236145 and 214653 is a Möbius annihilator.*

*Proof.* We first present the proof for the permutation 215463. Let  $\alpha = 215463$ ,  $\beta = \alpha_1[\epsilon] = 14352$ ,  $\beta' = \alpha_6[\epsilon] = 21435$  and  $\gamma = \alpha_{1,6}[\epsilon, \epsilon] = 1324$ . From Figure 5.4 (left) we see that, after the removal of the annihilators  $\alpha_3[\epsilon]$ ,  $\alpha_4[\epsilon]$  and  $\alpha_5[\epsilon]$ , the interval  $[1, \alpha]$  becomes diamond-tipped with core  $(\beta, \beta', \gamma)$ . Hence by Facts 26 and 27 we have  $\mu[1, \alpha] = 0$ .

Let  $\pi$  be a permutation of the form  $\tau_i[\alpha]$  for some  $\tau$  and  $i \leq |\tau|$ . We will show, by induction on  $|\tau|$ , that  $\pi$  is a zero. The case  $|\tau| = 1$  has been proved in the previous paragraph.

Assume that  $|\tau| > 1$ . We will demonstrate that we can remove some zeros from the interval  $[1, \pi]$  to end up with a diamond-tipped interval with core  $(\tau_i[\beta], \tau_i[\beta'], \tau_i[\gamma])$ . Choose a  $\lambda \in [1, \pi]$ . We can then write  $\lambda$  as  $\lambda = \tau_j^*[\alpha^*]$  for some  $\tau^* \leq \tau$  and some (possibly empty)  $\alpha^* \leq \alpha$ , where  $\tau^*$  has an embedding into  $\tau$  mapping  $j$  to  $i$ .

If  $\alpha^*$  is an annihilator, then  $\lambda$  is a zero and can be removed. If  $\alpha^* = \alpha$ , then  $|\tau^*| < |\tau|$ , and by induction,  $\lambda$  is a zero and can be removed. In all the other cases, we have  $\alpha^* \leq \beta$  or  $\alpha^* \leq \beta'$ , and in particular,  $\lambda$  belongs to  $[1, \tau_i[\beta]] \cup [1, \tau_i[\beta']]$ .

Suppose now that  $\lambda$  is in  $[1, \tau_i[\beta]] \cap [1, \tau_i[\beta']]$  but not in  $[1, \tau_i[\gamma]]$ . Since  $\lambda \leq \tau_i[\beta]$ , we can write it as  $\lambda = \tau_j^L[\beta^L]$ , for some  $\tau^L \leq \tau$  and  $\beta^L \leq \beta$ , where  $\tau^L$  has an embedding into  $\tau$  mapping  $j$  to  $i$ . Since  $\lambda \not\leq \tau_i[\gamma]$ , we know that  $\beta^L \not\leq \gamma$ . This means that  $\beta^L \in [1, \beta] \setminus [1, \gamma] = \{14352, 3241, 1342, 231\}$ . Similarly,  $\lambda \in [1, \tau_i[\beta']] \setminus [1, \tau_i[\gamma]]$  means that  $\lambda$  can be written as  $\lambda = \tau_k^R[\beta^R]$ , with  $\beta^R \in \{21435, 2143\}$ . Since  $\lambda$  has an interval copy of  $\beta^L$  as well as an interval copy of  $\beta^R$ , Theorem 31 shows that  $\lambda$  is a zero if  $\beta^L \in \{1342, 231\}$ , and Theorem 38 shows that  $\lambda$  is a zero if  $\beta^L \in \{14352, 3241\}$  (using that 3241 is a diagonal reflection of 2431). Therefore  $\lambda$  can be removed.

After the removal described above,  $[1, \pi]$  is transformed into a diamond-tipped interval, showing that  $\pi$  is a zero.

The arguments for the other two permutations are completely analogous. For 236145 we have  $\alpha = 236145$ ,  $\beta = 25134$ ,  $\beta' = 23514$ ,  $\gamma = 2413$ ,  $\beta^L \in \{25134, 1423\}$

and  $\beta^R \in \{23514, 2314\}$ , and use Theorems 37, 39 and 40. For 214653 we have  $\alpha = 214653$ ,  $\beta = 13542$ ,  $\beta' = 2143$ ,  $\gamma = 132$ ,  $\beta^L \in \{13542, 2431, 1342, 231\}$  and  $\beta^R \in \{2143, 213\}$ , and use Theorems 31, 37 and 38.  $\square$

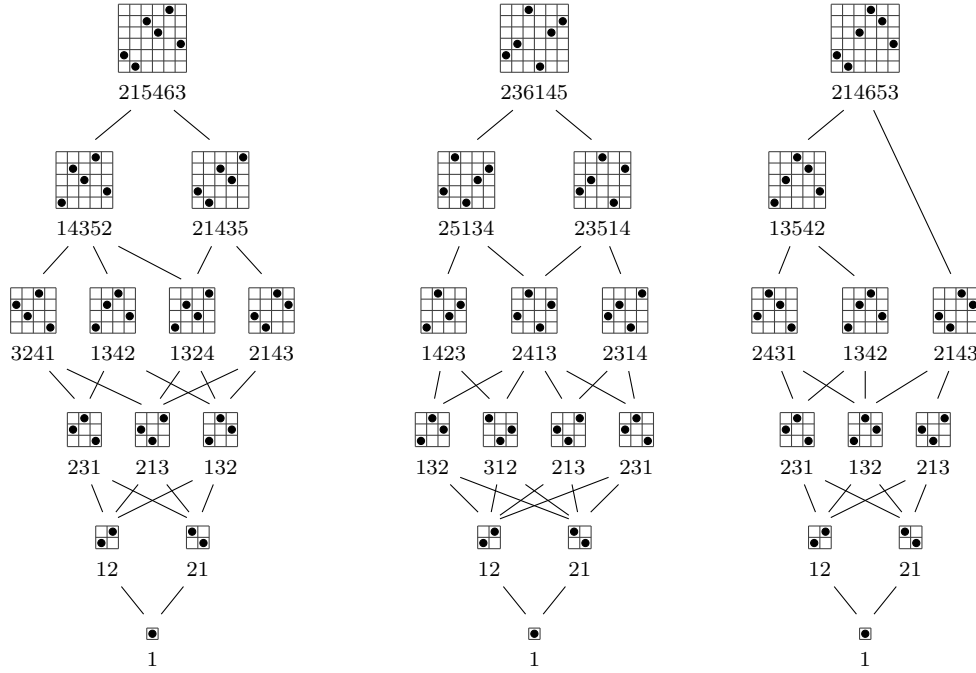


Figure 5.4: The three annihilators from Theorem 41, and the posets of their subpermutations. The figures omit the permutations with opposing adjacencies, as well as the permutations with an interval copy of a permutation of the form  $\alpha \oplus 1 \oplus \beta$ .

The annihilator 215463 of Theorem 41 can be written as a sum of two intervals, namely  $215463 = 21 \oplus 3241$ . One might wonder whether the two summands are in fact an annihilator pair. This, however, is not the case, as evidenced by the permutation  $32417685 = 3241 \oplus 3241$ , which is not a Möbius zero. An analogous example applies to  $214653 = 21 \oplus 2431$ .

In the proof of Theorem 41, it was crucial that for each  $\alpha \in \{215463, 236145, 214653\}$ , the interval  $[1, \alpha]$  becomes diamond-tipped after the removal of some annihilators. However, this property alone is not sufficient to make a permutation  $\alpha$  an annihilator. Consider, for instance, the permutation  $\alpha = 214635$ . We may routinely check that by removing some annihilators, the interval  $[1, \alpha]$  can be made diamond-tipped with

core  $(\beta = 13524, \beta' = 21435, \gamma = 1324)$ . This implies that  $\alpha$  is a Möbius zero by Facts 26 and 27; however, it does not imply that  $\alpha$  is an annihilator. In fact,  $\alpha$  is not an annihilator, as demonstrated by the permutation

$$\begin{aligned}\pi &= 582741936_{2,4,5}[\beta, \alpha, \beta'] \\ &= 9, 17, 19, 21, 18, 20, 2, 12, 11, 14, 16, 13, 15, 5, 4, 7, 6, 8, 1, 22, 3, 10,\end{aligned}$$

whose principal Möbius function is 1, not 0. This example also shows that not all Möbius zeros are annihilators.

In fact, among permutations of size at most 6, there are up to symmetry four Möbius zeros that are not annihilators. Apart from the permutation 214635 pointed out above, there are three more examples: 235614, 254613 and 465213. To see that these three permutations are not annihilators, it suffices to check that for any  $\alpha \in \{235614, 254613, 465213\}$ , the permutation  $24153_2[\alpha]$  has non-zero principal Möbius function. We verified, with the help of a computer, that all the Möbius zeros of size at most 6 that are not symmetries of the four examples above can be shown to be annihilators by our results. This data is available at <https://iuuk.mff.cuni.cz/~jelinek/mf/zeros.txt>.



## 5.8 Concluding remarks

Given Theorem 31, it is natural to wonder if we can find a similar result that applies to a permutation with multiple adjacencies, but no opposing adjacencies. One difficulty here is that there are permutations that have multiple adjacencies, and do not have opposing adjacencies, where the principal Möbius function value is non-zero. As an example, any permutation  $\pi = 2, 1, 4, 3, \dots, 2k, 2k - 1 = \oplus^k 21$  has  $\mu[\pi] = -1$  by the results of Burstein et al. [18, Corollary 3].

Let  $d_n$  be the “density of zeros” of the Möbius function, that is, the probability that  $\mu[\pi] = 0$  for a uniformly random permutation  $\pi$  of size  $n$ . The asymptotic behaviour of  $d_n$  is still elusive.

**Problem 42.** Does the limit  $\lim_{n \rightarrow \infty} d_n$  exist? And if it does, what is its value?

Corollary 36 implies that  $\liminf_{n \rightarrow \infty} d_n \geq (1 - 1/e)^2 \geq 0.3995$ . We have no upper bound on  $d_n$  apart from the trivial bound  $d_n \leq 1$ , but computational data suggest that simple permutations very often (though not always) have non-zero principal Möbius function. Since a random permutation is simple with probability approaching  $1/e^2$  [2], this would suggest that  $\limsup_{n \rightarrow \infty} d_n$  is at most  $1 - 1/e^2 \approx 0.8647$ .

$n$	$d_n$	$n$	$d_n$
1	0.0000	8	0.5942
2	0.0000	9	0.6019
3	0.3333	10	0.6040
4	0.4167	11	0.6034
5	0.4833	12	0.6021
6	0.5361	13	0.6006
7	0.5742		

Table 5.1: The density of Möbius zeros among permutations of length  $n$ , with  $n = 1, \dots, 13$ .

Table 5.1 lists the values of  $d_n$  for  $n = 1, \dots, 13$ . The values are based on data supplied by Smith [47] for  $1 \leq n \leq 9$ , and calculations performed by the author of this thesis. Data files with the values of the principal Möbius function for all

permutations of length twelve or less are available from <https://doi.org/10.21954/ou.rd.7171997.v2>. Based on this somewhat limited numeric evidence, we make the following conjecture:

**Conjecture 43.** The values  $d_n$  are bounded from above by 0.6040.

It is natural to look for further ways to identify Möbius zeros and Möbius annihilators. Characterizing all the Möbius zeros would be an ambitious goal, since  $\mu[\pi]$  might be zero as a result of “accidental” cancellations with no deeper structural significance for  $\pi$ .

An *annihilator multiset* is a multiset of permutations  $A = \{\alpha_1, \dots, \alpha_n\}$  such that any permutation  $\pi$  that contains disjoint interval copies of the permutations  $\alpha_1, \dots, \alpha_n$  has  $\mu[\pi] = 0$ .

If  $A = \{\alpha_1, \dots, \alpha_n\}$  and  $B = \{\beta_1, \dots, \beta_m\}$  are annihilator multisets, then we say that  $A$  *contains*  $B$  if  $A \neq B$  and we can find the elements of  $B$  as disjoint interval copies in the elements of  $A$ . An annihilator multiset  $A$  is *minimal* if there is no annihilator multiset contained in  $A$ .

Using Corollary 3 of [18], which implies  $\mu[\pi] = \mu[\pi \oplus \pi]$  for  $\pi \neq 1$ , it is simple to show that the permutations in a minimal annihilator multiset are, in fact, all distinct, and so we can refer to *minimal annihilator sets* of permutations.

**Problem 44.** Which permutations are Möbius annihilators? Are there infinitely many minimal annihilator sets that contain just one element, and are not of the form  $\alpha \oplus 1 \oplus \beta$ ?

It seems likely to us that the proofs of Theorems 37–40 might be extended to give several more annihilator pairs, such as (312, 235614). However, we do not see any general pattern in these examples yet.

**Problem 45.** Are there infinitely many minimal annihilator sets with two elements?

**Problem 46.** Are there any minimal annihilator sets with more than two elements?

## 5.9 Chapter summary

Prior to the publication of the paper on which this chapter is based, the proportion of permutations where we had a (computationally) simple way to determine the value of the principal Möbius function was, asymptotically, zero. This chapter presents two main results.

The first, Theorem 31, tells us that if a permutation has opposing adjacencies, then the value of the principal Möbius function is zero. It is possible to determine if a permutation has opposing adjacencies in time proportional to the length of the permutation, so this is a test that is simple to implement. This is potentially useful to anyone wanting to compute values of the principal Möbius function.

The second main result comes from Corollary 36. In essence this gives us that the proportion of permutations where the principal Möbius function is zero is at least 0.3995. From a computational perspective, this implies that there are significant benefits from using Theorem 31 when determining the value of the principal Möbius function, as we have a linear time algorithm which, asymptotically, gives a positive result for nearly 40% of permutations.

### 5.9.1 Improving the lower bound for the density of zeros, $d_n$

Conjecture 43 suggests, based on some rather limited numerical evidence, that the density of zeros,  $d_n$ , of the principal Möbius function is bounded above by 0.6040. We know, from Corollary 36, that asymptotically  $d_n$  is bounded below by 0.3995. One area for further research would be to consider if we can find better bounds on the behaviour of  $d_n$ . One difficulty with this is that we would need to find a result where the number of permutations covered by the result is proportional to  $n!$ , where  $n$  is the length of the permutation. Any relationship that was less than factorial (for example, exponential or polynomial) would mean that the proportion of permutations covered would be, asymptotically, zero.

In the concluding remarks above, we note that it is natural to wonder if we can find

Length	$= 0$	$\neq 0$
4	6	2
5	30	8
6	170	38
7	1154	212
8	8954	1502
9	78006	13088
10	757966	130066
11	8132206	1436296
12	95463532	17403612

Table 5.2: Number of permutations with non-opposing adjacencies, classified by the value of the principal Möbius function.

a similar result that applies where a permutation has multiple adjacencies, but no opposing adjacencies. Such a result would have to account for the permutations that have multiple non-opposing adjacencies where the principal Möbius function value is non-zero. Table 5.2 shows, for lengths  $4, \dots, 12$ , the number of permutations with multiple non-opposing adjacencies broken down by whether the value of the principal Möbius function is zero or not.

This suggests that it might be possible to find a result similar to Theorem 31 for some or all of these cases, although, as noted, any such result will clearly need some additional criteria that will exclude permutations that have a non-zero principal Möbius function value.

We remark that the non-opposing adjacency case may be important because the proportion of permutations of length  $n$  that have non-opposing adjacencies is, asymptotically, non-zero, as we show now.

**Theorem 47.** *The proportion of permutations that have non-opposing adjacencies is, asymptotically, bounded below by 0.1944.*

*Proof.* We find a lower bound by counting permutations that have non-opposing adjacencies.

We will need to use

**Theorem 48** (Albert, Atkinson and Klazar [2, Theorem 5]). *The number of simple permutations of length  $n$ ,  $S(n)$ , is given by*

$$S(n) = \frac{n!}{e^2} \left( 1 - \frac{1}{n} + \frac{2}{n(n-1)} + O(n^{-3}) \right).$$

Let  $Z'(n)$  be the proportion of permutations of length  $n$  that have non-opposing adjacencies. Let  $n \geq 6$  be an integer; and let  $k$  be an integer in the range  $2, \dots, \lfloor n/2 \rfloor$ . Let  $\sigma$  be a simple permutation with length  $n - k$ . We will count the number of ways we can inflate  $\sigma$  with  $k$  adjacencies to obtain a permutation with length  $n$  that has non-opposing adjacencies. We can choose the positions to inflate in  $\binom{n-k}{k}$  ways. The positions chosen can be inflated by either 12 or 21, so there are just 2 distinct inflations by adjacencies. It follows that the number of ways to inflate  $\sigma$  that result in a permutation with non-opposing adjacencies is given by

$$2 \binom{n-k}{k}.$$

Since we are inflating simple permutations, it follows from Albert and Atkinson [1, Proposition 2] that the inflations are unique.

For an inflation to contain a non-opposing adjacency, we need to inflate at least two points. Further, to obtain a permutation of length  $n$  by inflating with adjacencies we can, at most, inflate  $\lfloor n/2 \rfloor$  positions. Now using Theorem 48 we can say that

$$Z'(n) \geq \frac{1}{n!} \sum_{k=2}^{\lfloor n/2 \rfloor} S(n-k) 2 \binom{n-k}{k}.$$

Since we are only interested in the asymptotic behaviour of  $Z'(n)$ , we can assume that  $n > 20$ , and so we write

$$\begin{aligned} \lim_{n \rightarrow \infty} Z'(n) &\geq \lim_{n \rightarrow \infty} \frac{1}{n!} \sum_{k=2}^{\lfloor n/2 \rfloor} S(n-k) 2 \binom{n-k}{k} \\ &\geq \lim_{n \rightarrow \infty} \frac{1}{n!} \sum_{k=2}^{10} S(n-k) 2 \binom{n-k}{k} \end{aligned}$$

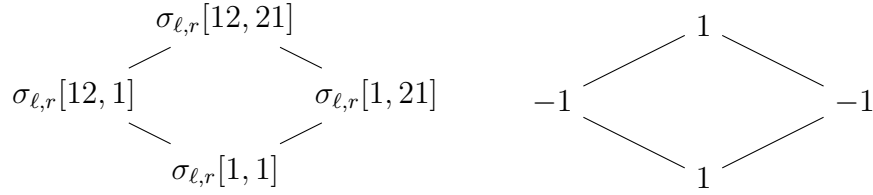


Figure 5.5: The Hasse diagram of the interval  $[\sigma, \sigma_{\ell,r}[12, 21]]$ , and the corresponding values of the Möbius function.

$$\geq 0.1944.$$

□

Thus if we could show that asymptotically, some fixed proportion of the permutations of length  $n$  with non-opposing adjacencies were all Möbius zeros, then we could improve the bounds given by Corollary 36.

### 5.9.2 Extending the “opposing adjacencies” theorem

It is natural to ask if we can extend Theorem 31 to handle cases where the lower bound of the interval is not 1. This is not possible in general, as if we take any permutation  $\sigma \neq 1$ , and inflate any two distinct points in positions  $\ell$  and  $r$  by 12 and 21 respectively, then  $\pi = \sigma_{\ell,r}[12, 21]$  has opposing adjacencies, but  $\mu[\sigma, \pi] = 1$ , as can be deduced from Figure 5.5.

Although we do not have a general extension of Theorem 31, we can show that:

**Theorem 49.** *If  $\sigma$  is adjacency-free, and  $\pi$  contains an interval order-isomorphic to a symmetry of 1243, then  $\mu[\sigma, \pi] = 0$ .*

*Proof.* First note that if  $\sigma \not\leq \pi$ , then  $\mu[\sigma, \pi] = 0$  from the definition of the Möbius function. Further, since  $\sigma$  is adjacency-free, we cannot have  $\sigma = \pi$ .

We can now assume that  $\sigma < \pi$ . Without loss of generality we can also assume, by symmetry, that the interval in  $\pi$  is order-isomorphic to 1243.

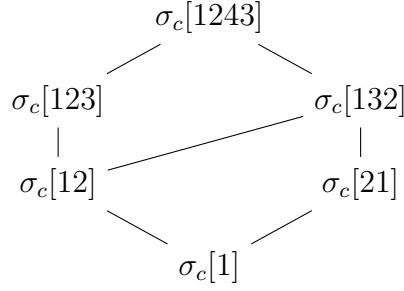


Figure 5.6: The Hasse diagram of the interval  $[\sigma, \sigma_c[1243]]$ .

We start by claiming that, for any permutation  $\sigma$  which is adjacency-free, and any  $c$  with  $1 \leq c \leq |\sigma|$ , we have  $\mu[\sigma, \sigma_c[1243]] = 0$ . The Hasse diagram of the interval  $[\sigma, \sigma_c[1243]]$  is shown in Figure 5.6. From the definition of the Möbius function, we have  $\mu[\sigma, \sigma_c[1]] = 1$ ,  $\mu[\sigma, \sigma_c[12]] = -1$ ,  $\mu[\sigma, \sigma_c[21]] = -1$ ,  $\mu[\sigma, \sigma_c[123]] = 0$ , and  $\mu[\sigma, \sigma_c[132]] = 1$ , and so  $\mu[\sigma, \sigma_c[1243]] = 0$ , and thus our claim is true.

Our argument now follows a similar pattern to that used by the proof of Theorem 31, and we restrict ourselves to highlighting the differences.

Assume that  $\pi$  is a proper inflation of  $\sigma$ , with length greater than  $|\sigma| + 4$ , and  $\pi$  contains an interval order-isomorphic to 1243. Let  $\gamma$  be the permutation formed by replacing an occurrence of 1243 in  $\pi$  by 12, so if  $\ell$  is the position of the first point of the 1243 selected, then  $\gamma_{\ell, \ell+1}[12, 21] = \pi$ . Let  $\lambda = \gamma_{\ell, \ell+1}[12, 1]$ , and let  $\rho = \gamma_{\ell, \ell+1}[1, 21]$ ; Define sets  $L = [\sigma, \lambda]$ ,  $R = [\sigma, \rho]$ ,  $G_\gamma = [\sigma, \gamma]$ ,  $G_x = L \cap R \setminus G_\gamma$ , and  $T = [\sigma, \pi] \setminus (L \cup R)$ .

Similarly to Theorem 31, we have

$$\mu[\sigma, \pi] = - \sum_{\tau \in L} \mu[\sigma, \tau] - \sum_{\tau \in R} \mu[\sigma, \tau] - \sum_{\tau \in T} \mu[\sigma, \tau] + \sum_{\tau \in G_\gamma} \mu[\sigma, \tau] + \sum_{\tau \in G_x} \mu[\sigma, \tau],$$

and the sums over the sets  $L$ ,  $R$  and  $G_\gamma$  are obviously zero. Using similar arguments to Theorem 31, we can see that every permutation  $\tau$  in  $T$  or  $G_x$  contains an interval order-isomorphic to 1243, and so by the inductive hypothesis, has  $\mu[\sigma, \tau] = 0$ , and thus we have  $\mu[\sigma, \pi] = 0$ .  $\square$

Although we cannot find a general extension to Theorem 31, we can find a necessary condition for a proper inflation of certain permutations to have a Möbius function value of zero. This is

**Lemma 50.** *If  $\sigma$  is adjacency-free, and  $\pi = \sigma[\alpha_1, \dots, \alpha_n]$  is a proper inflation of  $\sigma$ , then  $\mu[\sigma, \pi] = 0$  implies that at least one  $\alpha_i \notin \{1, 12, 21\}$ .*

*Proof.* Assume that every  $\alpha_i \in \{1, 12, 21\}$ . Let  $k$  be the number of  $\alpha_i$ -s that are not equal to 1, and let  $j_1, \dots, j_k$  be the indexes ( $i$ -s) where  $\alpha_i \neq 1$ , so  $\pi = \sigma_{j_1, \dots, j_k}[\alpha_{j_1}, \dots, \alpha_{j_k}]$ .

Then every permutation in the interval  $[\sigma, \pi]$  has a unique representation as  $\sigma_{j_1, \dots, j_k}[v_1, \dots, v_k]$ , where

$$v_i \in \begin{cases} \{1, 12\} & \text{if } \alpha_{j_i} = 12, \\ \{1, 21\} & \text{if } \alpha_{j_i} = 21. \end{cases}$$

So each position  $j_i$  can be inflated by one of two permutations, and thus there is an obvious isomorphism between permutations in the interval  $[\sigma, \pi]$  and binary numbers with  $k$  bits. It follows that the poset can be represented as a Boolean algebra, and so by a well-known result (see, for instance, Example 3.8.3 in Stanley [50]),  $\mu[\sigma, \pi] = (-1)^{|\pi| - |\sigma|}$ . Thus if  $\mu[\sigma, \pi] = 0$ , at least one  $\alpha_i \notin \{1, 12, 21\}$ .  $\square$





# Chapter 6

## 2413-balloons and the growth of the Möbius function

### 6.1 Preamble

This chapter is based on a published paper [\[31\]](#) which is sole work by the author.

In this chapter we show that the growth of the principal Möbius function on the permutation poset is exponential. This improves on previous work, which has shown that the growth is at least polynomial.

We define a method of constructing a permutation from a smaller permutation which we call “ballooning”. We show that if  $\beta$  is a 2413-balloon, and  $\pi$  is the 2413-balloon of  $\beta$ , then  $\mu[1, \pi] = 2\mu[1, \beta]$ . This allows us to construct a sequence of permutations  $\pi_1, \pi_2, \pi_3 \dots$ , with lengths  $n, n + 4, n + 8, \dots$  such that  $\mu[1, \pi_{i+1}] = 2\mu[1, \pi_i]$ , and this gives us exponential growth of the principal Möbius function. Further, our construction method gives permutations that lie within a hereditary class with finitely many simple permutations.

We also find an expression for the value of  $\mu[1, \pi]$ , where  $\pi$  is a 2413-balloon, with no restriction on the permutation being ballooned.

## 6.2 Introduction

In the concluding remarks to their seminal paper, Burstein, Jelínek, Jelínková and Steingrímsson [18] ask whether the principal Möbius function is unbounded, which is the first reference to the growth of the Möbius function in the literature. They show that if  $\pi$  is a separable permutation, then  $\mu[\pi] \in \{0, \pm 1\}$ , and thus is bounded. The separable permutations lie in a hereditary class which only contains the simple permutations 1, 12 and 21. They ask (Question 27) for which classes is  $\mu[\pi]$  bounded?

Smith [44] found an explicit case-wise formula for the principal Möbius function for all permutations with a single descent. For certain sets of permutations with a single descent, the associated formula is, up to a sign,

$$\mu[\pi] = \binom{k}{2},$$

where  $k$  is a linear expression in the length of  $\pi$ . This shows that the growth of the Möbius function is at least quadratic. Jelínek, Kantor, Kynčl and Tancer [25] show how to construct a sequence of permutations where the absolute value of the Möbius function grows according to the seventh power of the length.

We show that, given some permutation  $\beta$ , we can construct a permutation that we call the “2413-balloon” of  $\beta$ . This permutation will have four more points than  $\beta$ . We then show that if  $\pi$  is a 2413-balloon of  $\beta$ , and  $\beta$  is itself a 2413-balloon, then  $\mu[\pi] = 2\mu[\beta]$ . From this we deduce that the growth of the principal Möbius function is exponential. If  $\beta = 25314$  (which is a 2413-balloon), then we can construct a hereditary class that contains only the simple permutations  $\{1, 12, 21, 2413, 25314\}$ , where the growth of the principal Möbius function is exponential, answering questions in Burstein et al [18] and Jelínek et al [25].

We start by recalling some essential definitions and notation in Section 6.3, where we also provide some extensions of existing results. We formally define a 2413-balloon in Section 6.4, and we provide some results which will be used in the remainder of this chapter. In Section 6.5, we derive an expression for the value of  $\mu[\pi]$  when  $\pi$  is a double 2413-balloon, and following this we show that the growth of the Möbius

---

function is exponential in Section 6.6. We return to the topic of 2413-balloons in Section 6.7, and derive an expression for the value of  $\mu[\pi]$  when  $\pi$  is any 2413-balloon. Finally, we discuss the generalization of the balloon operator in Section 6.8. We also ask some questions regarding the growth of the Möbius function.

### 6.3 Essential definitions, notation, and results

In this section we recall some standard definitions and notation that we will use, and add some simple definitions and consequences of known results.

If  $\pi$  is a permutation with length  $n$ , then the number of *corners* of  $\pi$  is the number of points of  $\pi$  that are extremal in both position and value, that is,  $\pi_1 \in \{1, n\}$  or  $\pi_n \in \{1, n\}$ . It is easy to see that any permutation with length 2 or more can have at most two corners. We adopt the convention that the permutation 1 has one corner.

Recall that if a permutation  $\pi$  can be written as  $1 \oplus 1 \oplus \tau$ ,  $1 \ominus 1 \ominus \tau$ ,  $\tau \oplus 1 \oplus 1$ , or  $\tau \ominus 1 \ominus 1$ , where  $\tau$  is non-empty (so  $|\pi| \geq 3$ ), then we say that  $\pi$  has a *long corner*.

We now have

**Lemma 51.** *If  $\pi$  has a long corner, then  $\mu[\pi] = 0$ .*

**Lemma 52.** *If  $\pi$  can be written as  $\pi = 1 \oplus \tau$ , or  $\pi = \tau \oplus 1$  or  $\pi = 1 \ominus \tau$  or  $\pi = \tau \ominus 1$ , and does not have a long corner, then  $\mu[\pi] = -\mu[\tau]$ .*

These are well-known consequences of Propositions 1 and 2 of Burstein, Jelínek, Jelínková and Steingrímsson [18], and we refrain from providing proofs here. The reader is directed to Lemma 8 on page 40 in Chapter 4 for a proof of Lemma 51. Lemma 52 is a trivial extension of Corollary 3 in [18].

Recall that a triple adjacency is a monotonic interval of length 3. Smith shows that

**Lemma 53** (Smith [44, Lemma 1]). *If a permutation  $\pi$  contains a triple adjacency then  $\mu[\pi] = 0$ .*

A trivial corollary to Lemma 53 is

**Corollary 54.** *If a permutation contains a monotonic interval with length 3 or more, then  $\mu[\pi] = 0$ .*

Recall that Hall's Theorem [50, Proposition 3.8.5] says that

$$\mu[\sigma, \pi] = \sum_{c \in \mathcal{C}(\sigma, \pi)} (-1)^{|c|} = \sum_{i=1}^{|\pi|-1} (-1)^i K_i$$

where  $\mathcal{C}(\sigma, \pi)$  is the set of chains in the poset interval  $[\sigma, \pi]$  which contain both  $\sigma$  and  $\pi$ , and  $K_i$  is the number of chains of length  $i$ .

We can also use Hall's Theorem if we have a subset of chains that meet a specific criteria:

**Lemma 55.** *Let  $\pi$  be any permutation with length three or more. Let  $\psi$  be a permutation with  $1 < \psi < \pi$ . Let  $\mathcal{C}$  be the subset of chains in the poset interval  $[1, \pi]$  where the second-highest element is  $\psi$ . Then*

$$\sum_{c \in \mathcal{C}} (-1)^{|c|} = -\mu[\psi].$$

*Proof.* If we remove  $\pi$  from the chains in  $\mathcal{C}$ , then we have all of the chains in the poset interval  $[1, \psi]$ , and the Hall sum of these chains is, by definition,  $\mu[\psi]$ . It follows that the Hall sum of the chains in  $\mathcal{C}$  is  $-\mu[\psi]$ .  $\square$

**Corollary 56.** *Given a permutation  $\pi$ , and a set of permutations  $S$  where every  $\sigma \in S$  satisfies  $1 < \sigma < \pi$ , then if  $\mathcal{C}$  is the set of chains in the poset interval  $[1, \pi]$  where the second-highest element is in  $S$ , then the Hall sum of  $\mathcal{C}$  is  $-\sum_{\sigma \in S} \mu[\sigma]$ .*

*Proof.* First, partition  $\mathcal{C}$  based on the second-highest element, and then apply Lemma 55 to each partition.  $\square$

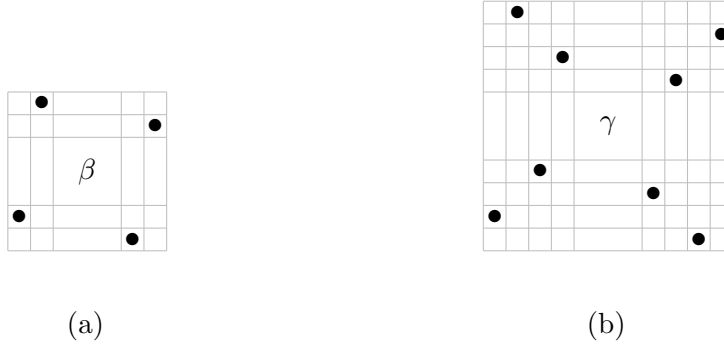


Figure 6.1: (a) The 2413-balloon  $2413 \odot \beta$  and (b) the double 2413-balloon  $2413 \odot 2413 \odot \gamma$ .

## 6.4 2413-Balloons

In this section we define the vocabulary and notation specific to this chapter. We also present some general results which will be used in later sections.

Given a non-empty permutation  $\beta$ , the *2413-balloon* of  $\beta$  is the permutation formed by inserting  $\beta$  into the centre of 2413, which we write as  $2413 \odot \beta$ . Formally, we have

$$(2413 \odot \beta)_i = \begin{cases} 2 & \text{if } i = 1 \\ |\beta| + 4 & \text{if } i = 2 \\ \beta_{i-2} + 2 & \text{if } i > 2 \text{ and } i \leq |\beta| + 2 \\ 1 & \text{if } i = |\beta| + 3 \\ |\beta| + 3 & \text{if } i = |\beta| + 4 \end{cases}$$

Figure 6.1(a) shows  $2413 \odot \beta$ . Throughout this chapter we will be discussing permutations that contain an interval copy of a smaller permutation. Examples of this smaller permutation are  $\beta$  in  $2413 \odot \beta$ , and  $\gamma$  in  $2413 \odot 2413 \odot \gamma$ , as shown in Figure 6.1. In figures where this is the case, the permutation plot scale will be non-linear so that the cell containing the interval copy ( $\beta$  and  $\gamma$  in our examples) is larger than the other cells.

The balloon operation as defined has to be right-associative and the definition given

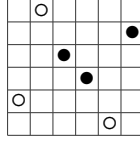


Figure 6.2: An embedding of  $213 = \overline{2413} \odot 21$  in  $264315 = 2413 \odot 21$ .

does not support overriding right-associativity. In other words,  $2413 \odot 2413 \odot \beta$  must be  $2413 \odot (2413 \odot \beta)$ , and  $(2413 \odot 2413) \odot \beta$  is not defined. In Section 6.8 we suggest how the balloon operation could be generalized.

Given some  $\pi = 2413 \odot \beta$ , if  $\beta$  is itself a 2413-balloon, so  $\pi = 2413 \odot 2413 \odot \gamma$ , then we say that  $\pi$  is a *double 2413-balloon*. Figure 6.1(b) shows a double 2413-balloon.

*Remark 57.* We can write  $2413 \odot \beta$  as the inflation  $25314[1, 1, \beta, 1, 1]$ . We refer the reader to Albert and Atkinson [1] for further details of inflations. In this chapter we use balloon notation, as we feel that this leads to a simpler exposition.

If we have  $\pi = 2413 \odot \beta$ , and we have some  $\sigma$  with  $\beta \leq \sigma < \pi$ , we will frequently want to represent  $\sigma$  in terms of sub-permutations of 2413 and the permutation  $\beta$ . We start by colouring the extremal points of  $\pi$  red, and all remaining points black. Note that the red points are a 2413 permutation, and the black points are  $\beta$ .

Now consider a specific embedding of  $\sigma$  into  $\pi$ , where we use all of the black points ( $\beta$ ). If the embedding is monochromatic ( $\sigma = \beta$ ) then we require no special notation. If the embedding is not monochromatic, then it must be the case that only some of the red points are used. We take 2413, and mark the red points that are unused with an overline, and then write  $\sigma$  using our balloon notation. As an example of this, if  $\pi = 2413 \odot 21 = 264315$ , and  $\sigma = 213$ , then we could represent  $\sigma$  as  $\overline{2413} \odot 21$ . This example is shown in Figure 6.2. We can see that if  $\beta \leq \sigma < 2413 \odot \beta$ , and  $\beta$  is not monotonic (i.e., not the identity permutation or its reverse), then there is a unique way to represent  $\sigma$  using this notation.

If we have  $\pi = 2413 \odot \beta$ , and  $\sigma$  is a permutation such that  $\beta \leq \sigma < \pi$ , then we say that  $\sigma$  is a *reduction* of  $\pi$ . If  $\sigma$  is a reduction of  $\pi = 2413 \odot \beta$ , and there is no  $\eta$  with  $|\eta| < |\beta|$  such that either  $\sigma$  is equal to  $2413 \odot \eta$ , or  $\sigma$  is a reduction of  $2413 \odot \eta$ ,



then we say that  $\sigma$  is a *proper reduction* of  $\pi$ . A reduction of  $\pi$  that is not a proper reduction is an *improper reduction*.

The following case-by-case analysis shows the improper reductions (of  $\pi$ ) based on the form of  $\beta$ .

- If  $\beta$  is a 2413-balloon, then  $\beta$  is the only improper reduction of  $\pi$ .
- If  $\beta$  is not a 2413-balloon, and  $\beta$  has no corners, then there are no improper reductions of  $\pi$ .
- If  $\beta$  has one corner, then there are four improper reductions of  $\pi$ . As an example, if  $\beta = 1 \oplus \gamma$ , then the improper reductions of  $\pi$  are  $\overline{2413} \odot \beta$ ,  $\overline{241\overline{3}} \odot \beta$ ,  $\overline{24\overline{13}} \odot \beta$ , and  $\beta$ .
- If  $\beta$  has two corners, then there are seven improper reductions of  $\pi$ . As an example, if  $\beta = 1 \oplus \gamma \oplus 1$ , then the improper reductions are  $\overline{2413} \odot \beta$ ,  $24\overline{1\overline{3}} \odot \beta$ ,  $\overline{241\overline{3}} \odot \beta$ ,  $\overline{24\overline{13}} \odot \beta$ ,  $\overline{241\overline{3}} \odot \beta$ ,  $\overline{24\overline{13}} \odot \beta$ , and  $\beta$ .

The set of permutations that are proper reductions of  $\pi$  is written as  $R_\pi$ . Figure 6.3 shows all the reductions (proper and improper) of  $\pi = 2413 \odot \beta$ .

The strategy that we will use in Sections 6.5 and 6.7 is to partition the chains in the poset interval  $[1, \pi]$  into three sets,  $\mathcal{R}$ ,  $\mathcal{G}$ , and  $\mathcal{B}$ . We then show that there are parity-reversing involutions on the sets  $\mathcal{G}$  and  $\mathcal{B}$ , and therefore, by Corollary 1, the Hall sum for each of these sets is zero, and so  $\mu[\pi]$  is given by the Hall sum of the set  $\mathcal{R}$ . Finally, we show that the Hall sum of  $\mathcal{R}$  can be written in terms of  $\mu[\beta]$ .

The chains in  $\mathcal{R}$  are those chains where the second-highest element is a proper reduction of  $\pi$ , so if  $\kappa_c$  is the second-highest element of a chain  $c$ , then  $c \in \mathcal{R}$  if and only if  $\kappa_c \in R_\pi$ . Note that, as mentioned earlier, the members of  $R_\pi$ , and hence the chains in  $\mathcal{R}$ , depend on the form of  $\pi$ . It is easy to see that for any permutation  $\sigma \in R_\pi$  we must have  $|\sigma| \geq |\beta|$ .

We have some results that are independent of  $R_\pi$ , and, once we have given some further definitions, we present these in the current section to avoid repetition.

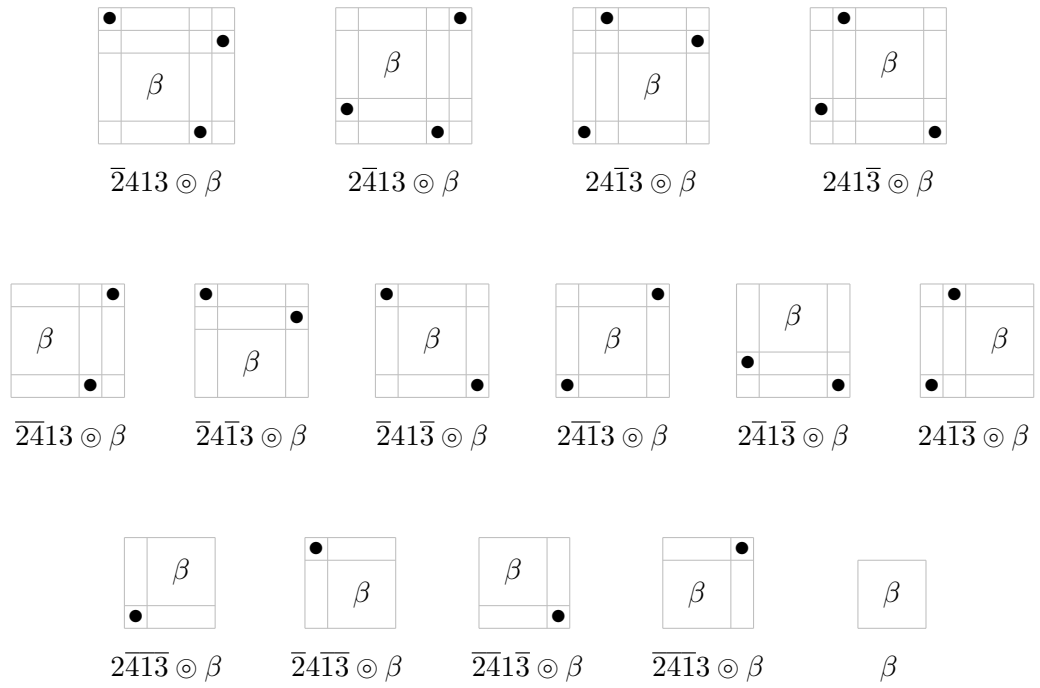


Figure 6.3: Reductions of  $\pi = 2413 \odot \beta$ . Some may not be proper reductions, depending on  $\beta$ .

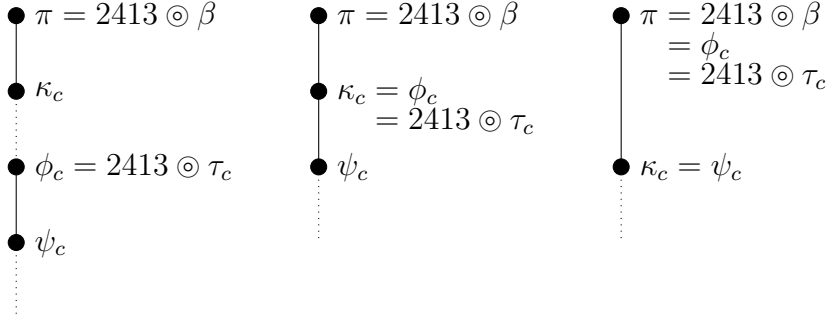


Figure 6.4: Examples of chains, showing some possible relationships between  $\pi$ ,  $\kappa_c$ ,  $\phi_c$ , and  $\psi_c$ .

Let  $\pi$  be a 2413-balloon, and let  $c$  be any chain in the poset interval  $[1, \pi]$ .

Since the top of the chain is, by definition, a 2413-balloon, it follows that  $c$  has a unique maximal segment that includes the element  $\pi$ , where every element in the segment is a 2413-balloon. We call the smallest element in this segment the *least 2413-balloon*<sup>1</sup>.

Further, since the permutation 1 is not a 2413-balloon, it follows that  $c$  has an element that is immediately below the least 2413-balloon in the chain, and we call this element the *pivot*.

We define  $\phi_c$  to be the least 2413-balloon in  $c$ ,  $\psi_c$  to be the pivot in  $c$ ,  $\tau_c$  to be the permutation that satisfies  $2413 \odot \tau_c = \phi_c$ , and  $\kappa_c$  to be the second-highest element of  $c$ . Note that  $\phi_c$  and  $\psi_c$  must be distinct, but we can have  $\tau_c = \psi_c$ . Further,  $\kappa_c$  is independent, and may be the same as  $\phi_c$ ,  $\psi_c$  or  $\tau_c$ . Figure 6.4 shows some example chains, highlighting these elements.

We are now in a position to give a definition of the sets  $\mathcal{R}$ ,  $\mathcal{G}$ , and  $\mathcal{B}$ . This definition depends on the set of proper reductions of  $\pi$ ,  $\mathcal{R}_\pi$ , which, as stated earlier, depends on the form of  $\beta$ .

Let  $\mathcal{C}$  be the set of chains in the poset interval  $[1, \pi]$ . We define subsets of  $\mathcal{C}$  as

<sup>1</sup>The name should really be “least 2413-balloon in the chain that has only 2413-balloons above it”.

$$\begin{aligned}\mathcal{R} &= \{c : c \in \mathcal{C} \text{ and } \kappa_c \in \mathbf{R}_\pi\}, \\ \mathcal{G} &= \{c : c \in \mathcal{C} \setminus \mathcal{R} \text{ and } \psi_c \leq 2413\}, \\ \mathcal{B} &= \{c : c \in \mathcal{C} \setminus (\mathcal{R} \cup \mathcal{G})\}.\end{aligned}$$

Given a pivot  $\psi_c$ , there is a unique permutation  $\eta_c$  which we call the *core* of  $\psi_c$ . In essence,  $\eta_c$  is the smallest permutation such that  $\psi_c < 2413 \odot \eta_c$ . To determine the core, we use the following algorithm:

Otherwise, if  $\psi_c$  can be written as  $(\eta \ominus 1) \oplus 1$  or  $1 \ominus (\eta \oplus 1)$   
or  $1 \ominus \eta \ominus 1$  or  $1 \oplus \eta \oplus 1$   
or  $(1 \oplus \eta) \ominus 1$  or  $1 \oplus (1 \ominus \eta)$ ,  
then set  $\eta_c = \eta$ .

Otherwise, if  $\psi_c$  can be written as  $1 \oplus \eta$  or  $1 \ominus \eta$  or  $\eta \ominus 1$  or  $\eta \oplus 1$ ,  
then set  $\eta_c = \eta$ .

Otherwise, set  $\eta_c = \psi_c$ .

We now define two functions, one for each of  $\mathcal{G}$  and  $\mathcal{B}$ , which will give us parity-reversing involutions.

$$\Phi_{\mathcal{G}}(c) = \begin{cases} c \setminus \{2413\} & \text{If } \psi_c = 2413 \\ c \cup \{2413\} & \text{If } \psi_c < 2413 \end{cases}$$

$$\Phi_{\mathcal{B}}(c) = \begin{cases} c \setminus \{2413 \odot \eta_c\} & \text{If } \eta_c = \tau_c \\ c \cup \{2413 \odot \eta_c\} & \text{If } \eta_c < \tau_c \end{cases}$$

*Remark 58.* If we were to allow the ballooning of the empty permutation  $\epsilon$ , and then treat 2413 as  $2413 \odot \epsilon$  then  $\Phi_{\mathcal{G}}(c)$  is subsumed by  $\Phi_{\mathcal{B}}(c)$ . Doing this, however, introduces additional complications in later proofs, and so we prefer two involutions.

For  $\Phi_{\mathcal{G}}(c)$  to be a parity-reversing involution on  $\mathcal{G}$ , we need to show that if  $c \in \mathcal{G}$ , then  $\Phi_{\mathcal{G}}(c)$  is a chain, that  $\Phi(c) \in \mathcal{G}$ , and that  $c$  and  $\Phi(c)$  have different parities. It is easy to see that this last condition is true. A similar comment applies to  $\Phi_{\mathcal{B}}(c)$  and  $\mathcal{B}$ .

For  $\Phi_{\mathcal{G}}(c)$  we can show that all the conditions hold for any  $R_{\pi}$ , regardless of the form of  $\beta$ . For  $\Phi_{\mathcal{B}}(c)$  we show that some weaker conditions hold for an arbitrary subset of the reductions of  $\pi$ , and then, when we have an explicit set of proper reductions, we show that all conditions hold. The following Lemma gives us a result that applies to  $\Phi_{\mathcal{G}}(c)$  and  $\Phi_{\mathcal{B}}(c)$  for any  $R_{\pi}$ , and we will use this result in both Section 6.5 and Section 6.7.

**Lemma 59.** *Let  $\pi = 2413 \odot \beta$ , with  $|\beta| > 4$ , and let  $\mathcal{R}$ ,  $\mathcal{G}$ , and  $\mathcal{B}$  be as defined above.*

- (a) *If  $c \in \mathcal{G}$ , then  $\Phi_{\mathcal{G}}(c) \in \mathcal{G}$ .*
- (b) *If  $c \in \mathcal{B}$ , with  $\eta_c = \tau_c$ , and  $\Phi_{\mathcal{B}}(c)$  is a chain, then  $\Phi_{\mathcal{B}}(c) \in \mathcal{B} \cup \mathcal{R}$ .*
- (c) *If  $c \in \mathcal{B}$ , with  $\eta_c < \tau_c$ , then  $\Phi_{\mathcal{B}}(c) \in \mathcal{B} \cup \mathcal{R}$ .*

*Proof. Case (a).* First, assume that  $c \in \mathcal{G}$  with  $\psi_c = 2413$ . Then  $c$  contains a segment  $2413 < 2413 \odot \tau_c$ , and  $c' = \Phi_{\mathcal{G}}(c) = c \setminus \{2413\}$ . We can see that  $c'$  is a chain,

as 2413 is neither the smallest nor the largest entry in  $c'$ . Further,  $\psi_{c'} < 2413$ . Since  $|\beta| > 4$ , and  $|\psi_{c'}| < 4$  we must have  $c' \notin \mathcal{R}$ , and therefore  $c' \in \mathcal{G}$ .

Now assume that  $c \in \mathcal{G}$  with  $\psi_c < 2413$ . Then  $c$  contains a segment  $\psi_c < 2413 \odot \tau_c$ , and  $c' = \Phi_{\mathcal{G}}(c) = c \cup \{2413\}$ . We can see that  $c'$  is a chain, since  $\psi_c < 2413 < 2413 \odot \tau_c$ , and further,  $\psi_{c'} = 2413$ . Since  $|\beta| > 4$ , and  $|\psi_{c'}| = 4$  we must have  $c' \notin \mathcal{R}$ , and therefore  $c' \in \mathcal{G}$ .

**Case (b).** Let  $c$  be a chain in  $\mathcal{B}$ , with  $\eta_c = \tau_c$ . Then  $c$  contains a segment  $\psi_c < 2413 \odot \tau_c$ , and  $c' = \Phi_{\mathcal{B}}(c) = c \setminus \{2413 \odot \tau_c\}$ . If  $\tau_c = \beta$ , then  $c'$  is not a chain, so we must have  $\tau_c < \beta$ , and therefore  $c'$  is a chain that contains a segment  $\psi_c < 2413 \odot \gamma$ , with  $\tau_c < \gamma$ . Now,  $\psi_c$  is the pivot of  $c'$ , so we cannot have  $c' \in \mathcal{G}$  as this would imply that  $c \in \mathcal{G}$ , which is a contradiction. Thus either  $c' \in \mathcal{R}$  or  $c' \in \mathcal{B}$ .

**Case (c).** Let  $c$  be a chain in  $\mathcal{B}$ , with  $\eta_c < \tau_c$ . Then  $c$  contains a segment  $\psi_c < 2413 \odot \tau_c$ , and  $c' = \Phi_{\mathcal{B}}(c) = c \cup \{2413 \odot \eta_c\}$ . We can see that  $c'$  is a chain since  $\psi_c < 2413 \odot \eta_c < 2413 \odot \tau_c$ . Now,  $\psi_c$  is the pivot of  $c'$ , so we cannot have  $c' \in \mathcal{G}$  as this would imply that  $c \in \mathcal{G}$ , which is a contradiction. So either  $c' \in \mathcal{R}$  or  $c' \in \mathcal{B}$ .  $\square$

We now have

**Observation 60.** *If  $\pi = 2413 \odot \beta$ , with  $|\beta| > 4$ , then to show that  $\Phi_{\mathcal{B}}$  is a parity-reversing involution on  $\mathcal{B}$  it is sufficient to show that:*

- (a) *If  $c \in \mathcal{B}$  and  $\eta_c = \tau_c$ , then  $\Phi_{\mathcal{B}}(c)$  is a chain, and  $\Phi_{\mathcal{B}}(c) \notin \mathcal{R}$ .*
- (b) *If  $c \in \mathcal{B}$ , and  $\eta_c < \tau_c$ , then  $\Phi_{\mathcal{B}}(c) \notin \mathcal{R}$ .*

*Further, if  $\Phi_{\mathcal{B}}$  is a parity-reversing involution on the set  $\mathcal{B}$ , then  $\mu[\pi] = -\sum_{\sigma \in \mathcal{R}_\pi} \mu[\sigma]$ .*

*Proof.* Combining (a) and (b) above with cases (b) and (c) of Lemma 59 gives us that  $\Phi_{\mathcal{B}}$  is a parity-reversing involution on  $\mathcal{B}$ .

This now gives us that  $\sum_{c \in \mathcal{B}} (-1)^{|c|} = 0$ . By Lemma 59, we know that  $\sum_{c \in \mathcal{G}} (-1)^{|c|} = 0$ , so we must have  $\mu[\pi] = \sum_{c \in \mathcal{R}} (-1)^{|c|}$ . Since the chains in  $\mathcal{R}$  are defined by the second-highest element ( $\kappa_c$ ) being in  $\mathcal{R}_\pi$ , the final part of the observation follows by applying Corollary 56.  $\square$

## 6.5 The principal Möbius function of double 2413-balloons

We are now able to state and prove our first major result.

**Theorem 61.** *Let  $\pi = 2413 \odot \beta$ , where  $\beta$  is a 2413-balloon, Then  $\mu[\pi] = 2\mu[\beta]$ .*

*Proof.* Note that  $\beta \notin \mathcal{R}_\pi$ , and further that  $|\beta| > 4$ , since  $\beta$  is a 2413-balloon.

Using Observation 60, we will show that  $\Phi_{\mathcal{B}}$  is a parity-reversing involution on  $\mathcal{B}$ . Once we have shown that we have parity-reversing involutions, we will then show how to express the Hall sum of  $\mathcal{R}$  in terms of  $\mu[\beta]$ .

*Proof that  $\Phi_{\mathcal{B}}$  is a parity-reversing involution on  $\mathcal{B}$ .* Let  $c$  be a chain in  $\mathcal{B}$ .

First, assume that  $\eta_c = \tau_c$ . If  $\tau_c = \beta$ , then either  $\psi_c$  is a proper reduction of  $\pi$ , or  $\psi_c = \beta$ . In the first case,  $c \in \mathcal{R}$ , and in the second case  $\psi_c$  is a 2413-balloon, and these are both contradictions. Thus we must have  $\tau_c < \beta$ , and so there is at least one permutation in  $c$  greater than  $\phi_c$ . It follows that  $c'$  is a chain. We now show that  $c' \notin \mathcal{R}$ . Assume, to the contrary, that  $c' \in \mathcal{R}$  which implies that  $\psi_c$  is a proper reduction of  $\pi$ . But now we have  $\eta_c = \beta$ , which is a contradiction, so  $\psi_c$  is not a proper reduction of  $\pi$ , therefore  $c' \notin \mathcal{R}$ .

Now assume that  $\eta_c < \tau_c$ . Let  $c' = \Phi_{\mathcal{B}}(c) = c \cup \{2413 \odot \eta_c\}$ . We know by Lemma 59 that this is a chain. Either  $\kappa_c = \kappa_{c'}$ , or  $\kappa_{c'}$  is a 2413-balloon. If  $\kappa_c = \kappa_{c'}$ , then  $c' \notin \mathcal{R}$ . If  $\kappa_{c'}$  is a 2413-balloon, then  $\kappa_{c'} \notin \mathcal{R}_\pi$ , so  $c' \notin \mathcal{R}$ . Thus we must have  $c' \notin \mathcal{R}$ .

So now we have that if  $c \in \mathcal{B}$  and  $\eta_c = \tau_c$ , then  $\Phi_{\mathcal{B}}(c)$  is a chain; and that for any  $c \in \mathcal{B}$ ,  $\Phi_{\mathcal{B}}(c) \in \mathcal{B}$ . It follows that  $\Phi_{\mathcal{B}}$  is a parity-reversing involution on  $\mathcal{B}$ .  $\square$

We now have that  $\Phi_{\mathcal{G}}$  and  $\Phi_{\mathcal{B}}$  are parity-reversing involutions on  $\mathcal{G}$  and  $\mathcal{B}$  respectively. It follows from Observation 60 that  $\mu[\pi] = -\sum_{\sigma \in \mathcal{R}_\pi} \mu[\sigma]$ . We now show how to express  $\mu[\sigma]$ , where  $\sigma \in \mathcal{R}_\pi$ , in terms of  $\mu[\beta]$ .

We start by noting that since  $\beta$  is a 2413-balloon, then  $\beta$  has no corners. Now, take the case where  $\sigma = \bar{2}413 \odot \beta$ , which is the first permutation in Figure 6.3. Note that we can write  $\sigma = 1 \ominus ((\beta \ominus 1) \oplus 1)$ . Applying Lemma 52 to the outermost three points in  $\sigma$  (those from the  $\bar{2}413$ ), we find that  $\mu[\sigma] = -\mu[\beta]$ . The other cases are similar, and this gives us:<sup>2</sup>

$\sigma$	$\mu[\sigma]$	$\sigma$	$\mu[\sigma]$	$\sigma$	$\mu[\sigma]$
$\bar{2}413 \odot \beta$	$-\mu[\beta]$	$\bar{2}413 \odot \beta$	$\mu[\beta]$	$241\bar{3} \odot \beta$	$-\mu[\beta]$
$2\bar{4}13 \odot \beta$	$-\mu[\beta]$	$\bar{2}4\bar{1}3 \odot \beta$	$\mu[\beta]$	$\bar{2}4\bar{1}\bar{3} \odot \beta$	$-\mu[\beta]$
$24\bar{1}3 \odot \beta$	$-\mu[\beta]$	$\bar{2}41\bar{3} \odot \beta$	$\mu[\beta]$	$\bar{2}41\bar{3} \odot \beta$	$-\mu[\beta]$
$241\bar{3} \odot \beta$	$-\mu[\beta]$	$2\bar{4}\bar{1}3 \odot \beta$	$\mu[\beta]$	$\bar{2}4\bar{1}3 \odot \beta$	$-\mu[\beta]$
		$2\bar{4}1\bar{3} \odot \beta$	$\mu[\beta]$		
		$24\bar{1}\bar{3} \odot \beta$	$\mu[\beta]$		

It is now easy to see that

$$\sum_{\sigma \in R_\pi} \mu[\sigma] = -2\mu[\beta]$$

and the result follows directly. □

---

<sup>2</sup>This table is slightly redundant, as the entries are determined by the parity of the red points. We include it as later results have similar tables where some values of  $\mu[\sigma]$  are zero, and this gives a consistent presentation.



## 6.6 The growth of the Möbius function

We define  $\text{AbsMax}_\mu(n) = \max\{|\mu[\pi]| : |\pi| = n\}$ . Previous work in [25] and [44] has shown that the growth of  $\text{AbsMax}_\mu(n)$  is at least polynomial. We will show that the growth is at least exponential. We have

**Theorem 62.** *For all  $n$ ,  $\text{AbsMax}_\mu(n) \geq 2^{\lfloor n/4 \rfloor - 1}$ .*

*Proof.* We start by defining a function to construct a permutation of length  $n$ .

$$\pi^{(n)} = \begin{cases} 1 & \text{If } n = 1 \\ 12 & \text{If } n = 2 \\ 132 & \text{If } n = 3 \\ 2413 & \text{If } n = 4 \\ 2413 \odot \pi^{(n-4)} & \text{Otherwise} \end{cases}$$

Note that for  $n > 8$ ,  $\pi^{(n)}$  is a double 2413-balloon. It is simple to calculate  $\mu[\pi^{(n)}]$  for  $n = 1, \dots, 8$ , and these values are given below.

$$\begin{array}{ll} \mu[\pi^{(1)}] = \mu[1] = 1, & \mu[\pi^{(5)}] = \mu[25314] = 4, \\ \mu[\pi^{(2)}] = \mu[12] = -1, & \mu[\pi^{(6)}] = \mu[263415] = -1, \\ \mu[\pi^{(3)}] = \mu[132] = 1, & \mu[\pi^{(7)}] = \mu[2735416] = 1, \\ \mu[\pi^{(4)}] = \mu[2413] = -3, & \mu[\pi^{(8)}] = \mu[28463517] = -6. \end{array}$$

These values match Theorem 62, and so this is true for  $n \leq 8$ . For  $n > 8$ ,  $\mu[\pi^{(n)}] = 2\mu[\pi^{(n-4)}]$  by Theorem 61, and the result follows immediately.  $\square$

*Remark 63.* It is easy to see that, with the definitions above, the only simple permutations that can be contained in  $\pi^{(n)}$  are 1, 12, 21, 2413, and 25314. This answers Problem 4.4 in [25], which asks whether  $\mu[\pi]$  is bounded on a hereditary class which contains only finitely many simple permutations, as, by Theorem 62, we have unbounded growth, but only finitely many simple permutations.

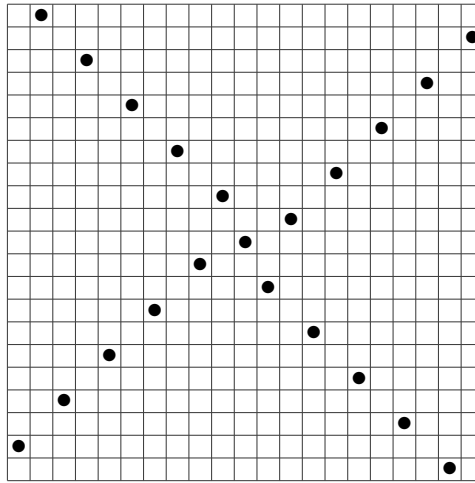


Figure 6.5: A permutation plot showing  $\pi^{(21)}$ .

If we repeat the ballooning process, as we do in  $\pi^{(n)}$ , then the permutation plot is rather striking. We illustrate this in Figure 6.5, which shows  $\pi^{(21)}$ .

## 6.7 The principal Möbius function of 2413-balloons

Theorem 61 gives us an expression for the value of the Möbius function  $\mu[\pi]$  when  $\pi$  is a double 2413-balloon. We expand on this to find an expression for the Möbius function  $\mu[\pi]$  when  $\pi$  is any 2413-balloon.

We start with a Lemma that handles the case where  $\beta$  is not a 2413-balloon, and has more than four points. The structure of our proof is similar to that of Theorem 61, but we present a complete argument to aid readability.

We will show

**Lemma 64.** *Let  $\pi = 2413 \odot \beta$ , where  $\beta$  is not a 2413-balloon, and  $|\beta| > 4$ . Then  $\mu[\pi] = \mu[\beta]$ .*

*Proof.* First note that if  $\beta$  is monotonic, then by Corollary 54 we have  $\mu[\beta] = 0 = \mu[\pi]$ . For the remainder of this proof, we assume that  $\beta$  is not monotonic.

If  $\beta$  has one corner, then without loss of generality, we can assume by symmetry that  $\beta = 1 \oplus \gamma$ . Similarly, if  $\beta$  has two corners, then we can assume that  $\beta = 1 \oplus \gamma \oplus 1$ .

As before, we will use Observation 60. We will show that  $\Phi_{\mathcal{B}}$  is a parity-reversing involution on  $\mathcal{B}$ . Once we have shown that we have parity-reversing involutions, we will then show how to express the Hall sum of  $\mathcal{R}$  in terms of  $\mu[\beta]$ .

The proper reductions of  $\pi$  depend on the number of corners of  $\beta$ . Below we list the improper reductions of  $\pi$  for each case.

Corners in $\beta$	Improper reductions of $\pi$
No corners	None.
One corner ( $\beta = 1 \oplus \gamma$ )	$\overline{2413} \odot \beta$ , $\overline{241\overline{3}} \odot \beta$ , $\overline{24\overline{1}3} \odot \beta$ , and $\beta$ .
Two corners ( $\beta = 1 \oplus \gamma \oplus 1$ )	$\overline{2413} \odot \beta$ , $24\overline{1\overline{3}} \odot \beta$ , $\overline{24\overline{1}3} \odot \beta$ , $\overline{241\overline{3}} \odot \beta$ , $\overline{241\overline{3}} \odot \beta$ , $\overline{24\overline{1}3} \odot \beta$ , and $\beta$ .

*Proof that  $\Phi_{\mathcal{B}}$  is a parity-reversing involution on  $\mathcal{B}$ .* Let  $c$  be a chain in  $\mathcal{B}$ .

First, assume that  $\eta_c = \tau_c$ , so  $c' = c \setminus \{2413 \odot \tau_c\}$ . We start by showing that  $c'$  is a valid chain. Assume otherwise, which implies  $\tau_c = \beta$ .

If  $\beta$  has no corners, then  $\psi_c \in R_\pi$ , so  $c \in \mathcal{R}$ , which is a contradiction.

If  $\beta$  has one corner, then either  $\psi_c \in R_\pi$ , which is a contradiction, or  $\psi_c \notin R_\pi$ . In the latter case, assume, without loss of generality, that  $\beta = 1 \oplus \gamma$ . Then  $\overline{2413} \odot \beta = \overline{2413} \odot \gamma$ ,  $\overline{241\overline{3}} \odot \beta = \overline{241\overline{3}} \odot \gamma$ ,  $\overline{2413} \odot \beta = \overline{2413} \odot \gamma$ , and  $\beta = \overline{241\overline{3}} \odot \gamma$ . Thus in all cases where  $\psi_c \notin R_\pi$ , we have that  $\eta_c$  is not minimal, which is a contradiction.

Finally, if  $\beta$  has two corners, then either  $\psi_c \in R_\pi$ , which is a contradiction, or  $\psi_c \notin R_\pi$ . The latter case implies that  $\psi_c = \beta$ , and then we have that either  $\psi_c = 1 \oplus \gamma \oplus 1 = \overline{2413} \odot \gamma$  or  $\psi_c = 1 \ominus \gamma \ominus 1 = \overline{241\overline{3}} \odot \gamma$ , so  $\eta_c$  is not minimal, which is a contradiction.

Thus we have that  $c'$  must be a chain, and, moreover,  $\tau_c \neq \beta$ .

We now show that  $c' \notin \mathcal{R}$ . Assume, to the contrary, that  $c' \in \mathcal{R}$  which implies that  $\psi_{c'}$  is a proper reduction of  $\pi$ . But now we have  $\eta_c = \beta$ , but this would give  $\tau_c = \beta$ , which is a contradiction, therefore  $c' \notin \mathcal{R}$ .

Now assume that  $\eta_c < \tau_c$ . Let  $c' = \Phi_{\mathcal{B}}(c) = c \cup \{2413 \odot \eta_c\}$ , and we know from Lemma 59 that  $c'$  is a chain. Now either  $\kappa_c = \kappa_{c'}$ , or  $\kappa_{c'} = 2413 \odot \eta_c$  is a 2413-balloon. In either case we have  $c' \notin \mathcal{R}$ .

So if  $c \in \mathcal{B}$ , then  $\Phi_{\mathcal{B}}(c)$  is a chain in  $\mathcal{B}$ , and thus  $\Phi_{\mathcal{B}}$  is a parity-reversing involution.  $\square$

We have shown that  $\Phi_{\mathcal{G}}$  and  $\Phi_{\mathcal{B}}$  are parity-reversing involutions on  $\mathcal{G}$  and  $\mathcal{B}$  respectively. It follows from Observation 60 that  $\mu[\pi] = -\sum_{\sigma \in R_\pi} \mu[\sigma]$ . We now show how to express  $\mu[\sigma]$ , where  $\sigma \in R_\pi$  in terms of  $\mu[\beta]$ . We use a similar mechanism to that used in Theorem 61. There are some additional considerations where  $\beta$  has one or two corners.

As an example, take the case where  $\sigma = \overline{2413} \odot \beta$ , and  $\beta$  has one corner, and so, by our assumption, can be written as  $1 \oplus \gamma$ . We can write  $\sigma = ((1 \oplus \beta) \ominus 1) \oplus 1$ ,

and expanding  $\beta$  we have  $\sigma = ((1 \oplus 1 \oplus \gamma) \ominus 1) \oplus 1$ , Applying Lemma 52 to the outermost two points in  $\sigma$ , we find that  $\mu[\sigma] = \mu[1 \oplus 1 \oplus \gamma]$ , and by Lemma 51 we now have  $\mu[\sigma] = 0$ . Because of this, our analysis depends on the number of corners of  $\beta$ , and we consider each case separately below.

If  $\beta$  has no corners, then we have

$\sigma$	$\mu[\sigma]$	$\sigma$	$\mu[\sigma]$	$\sigma$	$\mu[\sigma]$
$\bar{2}413 \odot \beta$	$-\mu[\beta]$	$\bar{2}413 \odot \beta$	$\mu[\beta]$	$241\bar{3} \odot \beta$	$-\mu[\beta]$
$2\bar{4}13 \odot \beta$	$-\mu[\beta]$	$\bar{2}4\bar{1}3 \odot \beta$	$\mu[\beta]$	$\bar{2}4\bar{1}3 \odot \beta$	$-\mu[\beta]$
$24\bar{1}3 \odot \beta$	$-\mu[\beta]$	$\bar{2}41\bar{3} \odot \beta$	$\mu[\beta]$	$\bar{2}41\bar{3} \odot \beta$	$-\mu[\beta]$
$241\bar{3} \odot \beta$	$-\mu[\beta]$	$24\bar{1}3 \odot \beta$	$\mu[\beta]$	$\bar{2}413 \odot \beta$	$-\mu[\beta]$
		$241\bar{3} \odot \beta$	$\mu[\beta]$		
		$24\bar{1}\bar{3} \odot \beta$	$\mu[\beta]$	$\beta$	$\mu[\beta]$

If  $\beta$  has one corner, under our assumption that  $\beta = 1 \oplus \gamma$ , we have

$\sigma$	$\mu[\sigma]$	$\sigma$	$\mu[\sigma]$	$\sigma$	$\mu[\sigma]$
$\bar{2}413 \odot \beta$	$-\mu[\beta]$	$\bar{2}4\bar{1}3 \odot \beta$	$\mu[\beta]$	$241\bar{3} \odot \beta$	0
$2\bar{4}13 \odot \beta$	0	$\bar{2}41\bar{3} \odot \beta$	$\mu[\beta]$	$\bar{2}41\bar{3} \odot \beta$	$-\mu[\beta]$
$24\bar{1}3 \odot \beta$	$-\mu[\beta]$	$24\bar{1}3 \odot \beta$	0		
$241\bar{3} \odot \beta$	$-\mu[\beta]$	$241\bar{3} \odot \beta$	0		
		$24\bar{1}\bar{3} \odot \beta$	$\mu[\beta]$		

Finally, if  $\beta$  has two corners, under our assumption that  $\beta = 1 \oplus \gamma \oplus 1$ , we have

$\sigma$	$\mu[\sigma]$	$\sigma$	$\mu[\sigma]$
$\bar{2}413 \odot \beta$	$-\mu[\beta]$	$\bar{2}4\bar{1}3 \odot \beta$	0
$2\bar{4}13 \odot \beta$	0	$\bar{2}41\bar{3} \odot \beta$	$\mu[\beta]$
$24\bar{1}3 \odot \beta$	0	$24\bar{1}3 \odot \beta$	0
$241\bar{3} \odot \beta$	$-\mu[\beta]$	$241\bar{3} \odot \beta$	0

In all three cases we have

$$\sum_{\sigma \in R_\pi} \mu[\sigma] = -\mu[\beta]$$

and the result follows directly.  $\square$

We are now in a position to state and prove the main Theorem for this section.

**Theorem 65.** *Let  $\pi = 2413 \odot \beta$ . Then*

$$\mu[\pi] = \begin{cases} 4 & \text{If } \beta = 1 \\ -6 & \text{If } \beta = 2413 \\ 2\mu[\beta] & \text{If } \beta \text{ is a 2413-balloon} \\ \mu[\beta] & \text{Otherwise.} \end{cases}$$

*Proof.* The value of  $\mu[2413 \odot \beta]$  for the symmetry classes of  $\beta$  with  $|\beta| \leq 4$  are shown below.

$\beta$	$\mu[\beta]$	$\mu[2413 \odot \beta]$	$\beta$	$\mu[\beta]$	$\mu[2413 \odot \beta]$
1	1	4	1324	-1	-1
12	-1	-1	1342	-1	-1
123	0	0	1432	0	0
132	1	1	2143	-1	-1
1234	0	0	2413	-3	-6
1243	0	0			

It is easy to see that these values meet Theorem 65. We now combine Theorem 61 and Lemma 64 to complete the proof.  $\square$

## 6.8 Concluding remarks

### 6.8.1 Generalising the balloon operator

Given two permutations  $\alpha$  and  $\beta$ , with lengths  $a$  and  $b$  respectively, and two integers  $i, j$  which satisfy  $0 \leq i, j \leq a$ , the  $i, j$ -balloon of  $\beta$  by  $\alpha$ , written as  $\alpha \odot_{i,j} \beta$ , is the permutation formed by inserting the permutation  $\beta$  into  $\alpha$  between the  $i$ -th and  $i + 1$ -th columns of  $\alpha$ , and between the  $j$ -th and  $j + 1$ -th rows of  $\alpha$ . The integers  $i$  and  $j$  are, collectively, the *indexes* of the balloon.

Formally, we have

$$(\alpha \odot_{i,j} \beta)_x = \begin{cases} \alpha_x & \text{if } x \leq i \text{ and } \alpha_x \leq j \\ \alpha_x + |\beta| & \text{if } x \leq i \text{ and } \alpha_x > j \\ \beta_{x-i} + j & \text{if } x > i \text{ and } x \leq i + |\beta| \\ \alpha_{x-|\beta|} & \text{if } x > i + |\beta| \text{ and } \alpha_{x-|\beta|} \leq j \\ \alpha_{x-|\beta|} + |\beta| & \text{if } x > i + |\beta| \text{ and } \alpha_{x-|\beta|} > j \end{cases}$$

As before, the balloon notation is not associative. Unlike 2413-balloons, which have to be interpreted as right-associative, generalized balloons can use brackets to define associativity. Note that the 2413-balloon defined in Section 6.3 is written as  $2413 \odot_{2,2} \beta$  in our generalized notation.

We remark that for any  $\alpha$  and any  $\beta$ , we have  $\alpha \odot_{0,0} \beta = \alpha \oplus \beta$ , and we can easily determine  $\mu[\alpha \oplus \beta]$  using results from Propositions 1 and 2 of Burstein, Jelínek, Jelínková and Steingrímsson [18].

### 6.8.2 Generalised 2413-balloons

If we restrict  $\alpha$  to 2413, then, up to symmetry, there are seven possible values for the indexes:  $(0, 0)$ ,  $(0, 1)$ ,  $(0, 2)$ ,  $(1, 0)$ ,  $(1, 1)$ ,  $(1, 2)$ , and  $(2, 2)$ . Theorem 65 handles

the case where the indexes are  $(2, 2)$ , and [18] handles the case where the indexes are  $(0, 0)$ . For the other indexes, we have

**Conjecture 66.** Let  $\pi = 2413 \odot_{i,j} \beta$ , where  $(i, j) \in \{(0, 1), (0, 2), (1, 1), (1, 2)\}$ . Then

$$\mu[\pi] = \begin{cases} 0 & \text{If } (i, j) = (0, 1) \text{ and } \beta = \tau \oplus 1 \\ 0 & \text{If } (i, j) = (0, 2) \text{ and } \beta = \tau \ominus 1 \\ 0 & \text{If } (i, j) = (1, 1) \text{ and } \beta = 1 \ominus \tau \text{ or } \beta = 12 \\ 0 & \text{If } (i, j) = (1, 2) \text{ and } \beta = 1 \oplus \tau \\ \mu[\beta] & \text{Otherwise.} \end{cases}$$

and

**Conjecture 67.** Let  $\pi = 2413 \odot_{1,0} \beta$ . Then

$$\mu[\pi] = \begin{cases} 6 & \text{If } \beta = 1 \\ -2 & \text{If } \beta = 21 \\ 0 & \text{If } \beta = 312 \\ 2\mu[\beta] & \text{If } \beta = 2413 \odot_{1,0} \gamma \\ \mu[\beta] & \text{Otherwise.} \end{cases}$$

Lemma 72 in Chapter 7 can be applied to the cases where the indexes are  $(0, 1)$ ,  $(0, 2)$ , or  $(1, 0)$ . Using this lemma, together with some of the techniques used earlier in this chapter, it is easy to show that Conjecture 66 is true in these cases. For brevity, we do not provide proofs here.

### 6.8.3 Bounding the Möbius function on hereditary classes

Corollary 24 in Burstein, Jelínek, Jelínková and Steingrímsson [18] gives us that if  $\pi$  is separable, then  $\mu[\pi] \in \{0, \pm 1\}$ . The simple permutations in the hereditary class of



separable permutations are 1, 12, and 21. In Remark 63 we have unbounded growth where the simple permutations in the hereditary class are just 1, 12, 21, 2413, and 25314, so adding 2413 and 25314 to the simple permutations moves us from bounded growth to unbounded growth. This then leads to:

**Question 68.** If  $C$  is a hereditary class containing just the simple permutations 1, 12, 21 and 2413, and  $\pi \in C$ , then is  $\mu[\pi]$  bounded? Further, if  $D$  is a hereditary class containing just the simple permutations 1, 12, 21, 2413, and 3142, and  $\pi \in D$ , then is  $\mu[\pi]$  bounded?

$n$	$\mu[\kappa_n]$	$\mu[\kappa_n]/\mu[\kappa_{n-1}]$
1	-1	-
2	-27	27
3	-117	4.333
4	-509	4.350
5	-2389	4.694
6	-10946	4.582
7	-51210	4.678

Table 6.1: Values of the principal Möbius function of  $\kappa_1, \dots, \kappa_7$ .

## 6.9 Chapter summary

The main result from this paper is a proof that the growth of  $\text{AbsMax}_\mu(n) = \max\{|\mu[\pi]| : |\pi| = n\}$  is at least exponential. This is proved by finding explicit recursions for the principal Möbius function of double 2413-balloons.

This result is not unexpected. Indeed, while the main result from Jelínek, Kantor, Kynčl and Tancer [25] is that the growth of  $\text{AbsMax}_\mu(n)$  is bounded below by an order-7 polynomial, in the final section of their paper they define a set of permutations  $\kappa_n$  as

$$\kappa_n = n + 1, n + 3, \dots, 3n - 1, 1, 3n + 1, 2, 3n + 2, \dots, n, 4n, n + 2, n + 4, \dots, 3n;$$

and then they conjecture that “the absolute value of  $\mu[\kappa_n]$  is exponential in  $n$ ”. The author computed the value of the principal Möbius function for  $\kappa_1, \dots, \kappa_7$ , and the results are shown in Table 6.1. Examining these values we can see two things:

- The ratio  $\mu[\kappa_n]/\mu[\kappa_{n-1}]$  is not an integer. In contrast, 2413 double-balloons grow by a factor of 2 at each iteration.
- $\kappa_n$  appears to grow at more than double the rate of a double 2413-balloon.

The author feels that it is very unlikely that  $\text{AbsMax}_\mu(n)$  is bounded below by something that grows faster than an exponential function.

The only mechanism currently available to determine  $\text{AbsMax}_\mu(n)$  is essentially to calculate the value of the principal Möbius function for every permutation of length  $n$ . While this has been done for lengths  $1, \dots, 13$ , the computational effort required to determine the value of  $\text{AbsMax}_\mu(14)$  is very large<sup>3</sup> and it is therefore unreasonable to expect this data to become available in the near future.

The technique used to construct 2413-balloon permutations can, as mentioned earlier, be thought of in terms of inflations of 25314. We commented earlier that we used balloon notation, as it is our belief that this gave us a simpler exposition. In Section 6.8 we generalized the ballooning process, and defined  $i, j$ -balloons. Although  $\alpha \odot_{i,j} \beta$  can be considered as an inflation of a permutation that is  $\alpha$  plus a single new point, we think that viewing permutations through the “balloon” lens gives a sufficiently different view that this technique could well have a wider application. Initial investigations by the author and others seem to indicate that most  $i, j$ -balloons behave “regularly”. Typically we find that  $\mu[\alpha \odot_{i,j} \beta]$  is a multiple of  $\mu[\beta]$ .

---

<sup>3</sup>The author estimates the effort on his HEDT PC would be around half a million CPU hours.

# Chapter 7

## The principal Möbius function of balloon permutations

### 7.1 Preamble

This chapter is based on independent research by the author that is, at the time of writing, still in progress. The author intends that this chapter will form the basis of work that will be submitted for publication, and at present this will be a single-author paper.

Generalised balloon permutations were introduced in Section 6.8 of Chapter 6. In this chapter we drop the “generalised” qualifier. A balloon permutation is formed from the merge of two permutations,  $\alpha$  and  $\beta$ , and has the property that  $\beta$  occurs as an interval copy in the balloon permutation.

In this chapter we find an expression for the principal Möbius function of balloon permutations in terms of a sum over a set of permutations, plus a correction factor.

We then show that for certain types of balloon permutation (“wedge permutations”) the correction factor is always zero. Further, we show that the principal Möbius function of a wedge permutation is always a multiple of the principal Möbius function of  $\beta$ .

## 7.2 Introduction

In this chapter we recall the definition of a balloon permutation from Chapter 6, and provide a second, equivalent, definition.

We show how, given two permutations  $\alpha$  and  $\beta$  we can construct a balloon permutation  $\alpha \odot_{i,j} \beta$ . We then provide some examples of balloon permutations, which include direct sums and skew sums, and introduce “block diagrams”, which show how  $\alpha$  and  $\beta$  are related.

We discuss how to represent permutations contained in a balloon permutation in terms of  $\alpha$  and  $\beta$ . This includes discussing how to resolve ambiguities that arise when a permutation contained in a balloon permutation has multiple embeddings.

We show that the principal Möbius function of  $\alpha \odot_{i,j} \beta$  can be expressed as a sum of the principal Möbius function over a certain subset of permutations contained in  $\alpha \odot_{i,j} \beta$ , plus a correction factor that is calculated from a specific set of chains. The subset of permutations has the property that they all contain  $\beta$  as an interval copy, although we note that not every permutation that contains  $\beta$  as an interval copy is included in the set of permutations.

One of the types of balloon permutation is a wedge permutation. We show that the correction factor for a wedge permutation is always zero. We further show that the value of the principal Möbius function of a wedge permutation is a multiple of the principal Möbius function of  $\beta$ .

We conclude this chapter with a brief summary of the results found. We briefly discuss certain phenomena which have been observed when the wedge operation is iterated.

## 7.3 Definitions, examples and notation

### 7.3.1 Balloon permutations

Generalised balloon permutations were introduced in Chapter 6, and we recall that they are defined as

$$(\alpha \odot_{i,j} \beta)_x = \begin{cases} \alpha_x & \text{if } x \leq i \text{ and } \alpha_x \leq j \\ \alpha_x + |\beta| & \text{if } x \leq i \text{ and } \alpha_x > j \\ \beta_{x-i} + j & \text{if } x > i \text{ and } x \leq i + |\beta| \\ \alpha_{x-|\beta|} & \text{if } x > i + |\beta| \text{ and } \alpha_{x-|\beta|} \leq j \\ \alpha_{x-|\beta|} + |\beta| & \text{if } x > i + |\beta| \text{ and } \alpha_{x-|\beta|} > j \end{cases} \quad (7.1)$$

with  $0 \leq i, j \leq |\alpha|$ .

If  $\alpha = 1$ , then  $\alpha \odot_{i,j} \beta$  is one of  $1 \oplus \beta$ ,  $1 \ominus \beta$ ,  $\beta \oplus 1$ , or  $\beta \ominus 1$ . In these cases it is trivial to find  $\mu[\alpha \odot_{i,j} \beta]$  using results from Burstein, Jelínek, Jelínková and Steingrímsson [18], and we exclude these cases from further consideration in this chapter by requiring that  $|\alpha| > 1$ .

Given two permutations,  $\alpha$  and  $\beta$ , we say that some permutation  $\pi$  is a *merge* of  $\alpha$  and  $\beta$  if the points of  $\pi$  can be coloured red or blue so that the red points are order-isomorphic to  $\alpha$ , and the blue points are order-isomorphic to  $\beta$ .

We can alternatively define a *balloon* permutation as the merge of two non-empty permutations  $\alpha$  and  $\beta$ , which we write as  $\alpha \odot_{i,j} \beta$ , with the additional requirement that the blue points (order-isomorphic to  $\beta$ ) must be an interval copy of  $\beta$ . One consequence of this last requirement is that

$$\alpha \odot_{i,j} \eta < \alpha \odot_{i,j} \zeta \quad \text{if and only if} \quad \eta < \zeta.$$

We call this the *nesting* condition.

Our alternative definition of a balloon permutation is somewhat abstract, and, as

we will see, certain values of  $i$  and/or  $j$  make a significant difference to our results. Before we continue with our discussion of notation, we first present several varieties of balloon permutations, together with a diagrammatic way of understanding how balloon permutations are structured.

### 7.3.2 Types of balloon permutations

When discussing types of balloon permutations  $\alpha \odot_{i,j} \beta$ , we will generally give a description of how  $\alpha$  and  $\beta$  are merged. We will also provide what we term a *block diagram*. A block diagram is used to give a visual representation of the merge, showing the relative locations of  $\alpha$  and  $\beta$  in  $\alpha \odot_{i,j} \beta$ . In all the examples we show, there are parts of the permutation plot of  $\alpha \odot_{i,j} \beta$  that are guaranteed to be empty. In a block diagram, we highlight these empty regions by shading them.

#### Balloon permutations

Figure 7.1 shows the block diagram for any balloon permutation. Note that, by

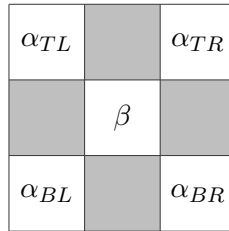


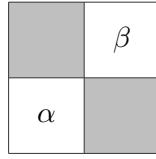
Figure 7.1: Block diagram for the generalised balloon permutation  $\alpha \odot_{i,j} \beta$ .

design, a block diagram does not include the indices used (the  $i$  and  $j$  in  $\alpha \odot_{i,j} \beta$ ), as the purpose of a block diagram is to provide a high-level view of the construction.

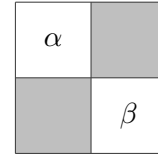
There are two sub-types of balloon permutations, which we describe now.

### Direct sums and skew sums

The simplest examples of balloon permutations, which have already appeared extensively in the literature, are the direct sum of two permutations,  $\alpha \oplus \beta$ , and the skew sum,  $\alpha \ominus \beta$ . Figure 7.2 shows the block diagram for direct sums and skew sums. Direct sums occur when we have  $\alpha \odot_{0,0} \beta$  or  $\alpha \odot_{|\alpha|,|\alpha|} \beta$ . Skew sums occur when we have  $\alpha \odot_{0,|\alpha|} \beta$  or  $\alpha \odot_{|\alpha|,0} \beta$ .



(a)  $\alpha \oplus \beta = \alpha \odot_{|\alpha|,|\alpha|} \beta$ .



(b)  $\alpha \ominus \beta = \alpha \odot_{|\alpha|,0} \beta$ .

Figure 7.2: Block diagrams for direct and skew sums.

### Wedge permutations

Our second type of balloon permutation is the *wedge permutation*. Wedge permutations occur when we have  $\alpha \odot_{i,0} \beta$  or  $\alpha \odot_{i,|\alpha|} \beta$  or  $\alpha \odot_{0,j} \beta$  or  $\alpha \odot_{|\alpha|,j} \beta$ . There are thus four (symmetric) ways in which we can construct a wedge permutation. For our purposes, we only need consider one symmetry, and we choose the version defined by  $\alpha \odot_{i,|\alpha|} \beta$ . We write this as  $\alpha \triangle_i \beta$ . Henceforth, we will refer to this as a wedge permutation without qualification.

Note that if  $i \in \{0, |\alpha|\}$ , then  $\alpha \triangle_i \beta$  is a direct or skew sum of  $\alpha$  and  $\beta$ . We will occasionally want to refer to wedge permutations that are not direct or skew sums, and we call these *proper wedge permutations*.

Another way to understand the construction of wedge permutations is to define  $\alpha \triangle_k \beta$  as the permutation formed by taking the first  $k$  points of  $\alpha$ , then appending all of the points from  $\beta$ , with their values increased by  $|\alpha|$ , and finally appending the remaining points of  $\alpha$ . Figure 7.3 shows the block diagram for a wedge permutation,



where  $\alpha_L$  represents the first  $k$  points for  $\alpha$ , and  $\alpha_R$  represents the remaining points of  $\alpha$ .

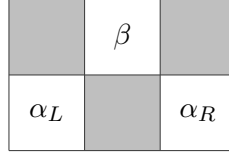


Figure 7.3: Block diagram for the wedge permutation  $\alpha \Delta_k \beta$ .

### 7.3.3 Notation

We have said that a balloon permutation will be written as  $\alpha \odot_{i,j} \beta$ . We now consider how we will represent some permutation  $\sigma$ , where  $\sigma \leq \alpha \odot_{i,j} \beta$ . Our aim is to define a notation where most permutations contained in  $\alpha \odot_{i,j} \beta$  have a unique representation.

Since  $\alpha \odot_{i,j} \beta$  is a merge of  $\alpha$  and  $\beta$ , we can colour the points of  $\alpha$  red, and the points of  $\beta$  blue. We note that for a merge in general there may be several possible colourings. By contrast, there is only one possible colouring for a balloon permutation. This is because  $\beta$  is an interval at a fixed position within  $\alpha \odot_{i,j} \beta$ , and so we know that the first point of  $\beta$  is in column  $i + 1$ , and the lowest point of  $\beta$  is in row  $j + 1$ . Since  $\beta$  is an interval, this then means that there is only one possible choice for the blue points, and so we have a unique colouring.

When we discuss permutations contained in  $\alpha \odot_{i,j} \beta$ , we will sometimes want to discuss how these permutations can be found as embeddings. Recall that if  $\sigma$  is contained in  $\pi$ , then an embedding of  $\sigma$  in  $\pi$  is a set of points of  $\pi$ , with cardinality  $|\sigma|$ , that is order-isomorphic to  $\sigma$ . Further, note that an embedding is not necessarily unique. As a (trivial) example, if  $|\pi| = n$ , then there are  $n$  distinct embeddings of the permutation 1 in  $\pi$ .

Now let  $\pi = \alpha \odot_{i,j} \beta$ , and assume that we have a permutation  $\sigma < \pi$ , and we want to describe an embedding  $\omega$  of  $\sigma$  in  $\pi$  in terms of the points of  $\alpha$  and  $\beta$ . The points of  $\omega$  in  $\alpha \odot_{i,j} \beta$  can be partitioned into two sets – those that are red, and those that

are blue. The blue points used in  $\omega$  will form a permutation  $\eta$ , where we allow  $\eta$  to be the empty permutation  $\epsilon$ . In general, we will not be concerned with exactly which red points are used. We know that the red points are a (possibly improper or empty) subset of the points of  $\alpha$ . We represent the embedding as

$$\omega = \bar{\alpha} \odot_{i,j} \eta.$$

This representation must be thought of as first finding the complete permutation  $\alpha \odot_{i,j} \eta$ , where  $\eta$  uses the blue points chosen, and then removing some or possibly all of the red points. With this notation, we can clearly understand the permutation formed by the blue points, as this is  $\eta$ , but we do not know which red points have been used. This notation does not distinguish between two embeddings where the blue points of each embedding are equivalent to the same permutation. This is deliberate, as when we consider embeddings, we will only be concerned with the permutation formed by the blue points, not exactly which blue points are used, and we think that the ambiguity in notation is more than offset by the increase in clarity in the discourse.

We now partition the permutations contained in  $\pi = \alpha \odot_{i,j} \beta$  into four subsets.

### Complete permutations

Our first set of permutations are those where there is an embedding that uses all of the red points. If  $\sigma$  is such a permutation, then it is possible to write  $\sigma = \alpha \odot_{i,j} \eta$  for some (possibly empty) permutation  $\eta$ . The permutation  $\eta$  is unique, since if we had  $\sigma = \alpha \odot_{i,j} \eta = \alpha \odot_{i,j} \zeta$ , then by the nesting condition we must have  $\eta = \zeta$ . We call these permutations *complete*, as they have an embedding which uses the complete set of red points. We will always write them in the form  $\alpha \odot_{i,j} \eta$ . The permutation  $\pi = \alpha \odot_{i,j} \beta$  is, of course, complete.

### Proper reductions

Our second set of permutations are those where every embedding uses all of the blue points, excluding the permutation  $\pi$ , which, as noted above, is complete. For these permutations, we will not be interested in understanding exactly which red points are used in any embedding. We write these permutations in the form  $\bar{\alpha} \odot_{i,j} \beta$ . This representation must be thought of as first finding the complete permutation  $\alpha \odot_{i,j} \beta$ , and then removing some or possibly all of the red points. We call these permutations *proper reductions*. Given  $\pi = \alpha \odot_{i,j} \beta$ , we denote the set of permutations that are proper reductions as  $R_\pi$ . Note that no proper reduction can be complete.

### Matryoshka permutations

Our third set of permutations are a subset of the permutations that are neither complete, nor proper reductions, and satisfy a specific condition (the “matryoshka” condition).

Given any two permutations  $\sigma$  and  $\pi$ , there is a set of all possible embeddings of  $\sigma$  into  $\pi$ , which we write  $\sigma(\pi)$ . Since we are only interested in cases where  $\pi = \alpha \odot_{i,j} \beta$ , and where  $\sigma$  is neither complete, nor a proper reduction, we can extend our notation to write

$$\sigma(\pi) = \{\bar{\alpha}^1 \odot_{i,j} \eta^1, \dots, \bar{\alpha}^n \odot_{i,j} \eta^n\}.$$

Note that there may be cases where there are two embeddings  $\bar{\alpha}^k \odot_{i,j} \eta^k$ , and  $\bar{\alpha}^\ell \odot_{i,j} \eta^\ell$ , with  $k \neq \ell$ , where the permutations  $\eta^k$  and  $\eta^\ell$  are the same. We are interested in understanding which permutations occur as  $\eta$  in  $\bar{\alpha} \odot_{i,j} \eta$  in the set of embeddings  $\sigma(\pi)$ , so our next step is to form the set of permutations that occur as some  $\eta$  in  $\sigma(\pi)$ . We define

$$E_{\sigma(\pi)} = \{\zeta : \sigma(\pi) \text{ contains an embedding } \bar{\alpha} \odot_{i,j} \zeta\}.$$

and, given some  $E_{\sigma(\pi)}$ , we label the elements as  $E_{\sigma(\pi)} = \{\zeta_1, \dots, \zeta_m\}$ . Note that  $E_{\sigma(\pi)}$  is a set of permutations, and that this set can include the empty permutation  $\epsilon$ . Now, if there is an integer  $k$ , with  $1 \leq k \leq m$  such that for all  $\ell = 1, \dots, m$ , and

$\ell \neq k$  we have  $\zeta_k < \zeta_\ell$ , then we say that the permutation  $\sigma$  is *matryoshka*, and we will write these permutations in the form  $\bar{\alpha} \odot_{i,j} \zeta$ , where  $\zeta = \zeta_k$ . As before, this representation must be thought of as first finding the complete permutation  $\alpha \odot_{i,j} \zeta$ , and then removing some or possibly all of the red points. The set of matryoshka permutations forms our third set.

If a permutation  $\sigma$  is not complete, and is not a proper reduction, and has an embedding  $\bar{\alpha} \odot_{i,j} \eta$ , where  $\eta \in \{\epsilon, 1\}$ , then it is easy to see that  $\sigma$  must be matryoshka.

### Defective permutations

The remaining permutations are not complete permutations, proper reductions, or matryoshka. We say that these permutations are *defective*. We will not need to concern ourselves with a unique representation of a defective permutation.

### Notation for elements of a chain

Our main arguments will be based on partitioning the chains in the poset, and then applying Corollary 1. Given  $\pi = \alpha \odot_{i,j} \beta$ , we now introduce the terminology and notation we will use in handling the chains in the poset  $[1, \pi]$ .

Let  $c$  be a chain in the interval  $[1, \pi]$ . We start by noting that the top element of every chain in the interval  $[1, \pi]$  is  $\pi = \alpha \odot_{i,j} \beta$ , and the bottom element is the permutation 1. Recall that we have  $|\alpha| > 1$ , so the permutation 1 cannot be written as  $\alpha \odot_{i,j} \eta$  for any  $\eta$ . It follows that every chain contains a largest permutation that cannot be written as  $\alpha \odot_{i,j} \eta$ . Note that since we cannot write this permutation as  $\alpha \odot_{i,j} \eta$  for any  $\eta$ , this permutation is not complete. As in Chapter 6, we call this permutation the *pivot*, and denote it by  $\psi_c$ . Since  $\psi_c$  is not complete, and the highest element of the chain,  $\pi$ , is complete, there must be a permutation above  $\psi_c$  in the chain. We call the permutation above  $\psi_c$  in the chain  $\phi_c$ . Finally, since every chain has at least two elements, there is always a second-highest permutation in the chain, and we call this  $\kappa_c$ .

We remark that  $\psi_c$  cannot be a complete permutation, and that  $\phi_c$ , and every permutation above  $\phi_c$  in the chain, must be a complete permutation.

We will partition the chains in the poset into three sets. The first set,  $\mathcal{R}$ , consists of those chains where  $\kappa_c$  is a proper reduction. The second set,  $\mathcal{M}$ , comprises chains  $c$  that are not in  $\mathcal{R}$ , where  $\psi_c$  is matryoshka. Our final set  $\mathcal{G}$  contains the remaining chains.

Formally, we define

$$\begin{aligned}
 \mathcal{C} &= \text{The set of all chains in } [1, \pi] \\
 \mathcal{R} &= \{c : c \in \mathcal{C}; \kappa_c \text{ is a proper reduction}\} \\
 \mathcal{M} &= \{c : c \in \mathcal{C} \setminus \mathcal{R}; \psi_c \text{ is matryoshka}\} \\
 \mathcal{G} &= \{c : c \in \mathcal{C} \setminus (\mathcal{R} \cup \mathcal{M})\}.
 \end{aligned} \tag{7.2}$$

We now have all the terminology and notation to prove our first result in this chapter.

## 7.4 The principal Möbius function of balloon permutations

Let  $\pi = \alpha \odot_{i,j} \beta$ . Our aim in this section is to derive an expression for  $\mu[\pi]$  as follows.

**Theorem 69.** *If  $\pi = \alpha \odot_{i,j} \beta$ , and  $\mathcal{G}$  is as defined in Equation 7.2, then*

$$\mu[\pi] = - \sum_{\lambda \in \mathcal{R}_\pi} \mu[\lambda] + \sum_{c \in \mathcal{G}} (-1)^{|c|}.$$

*Proof.* Since the sets  $\mathcal{R}$ ,  $\mathcal{M}$ , and  $\mathcal{G}$  partition the chains in the poset, we can write

$$\mu[\pi] = \sum_{c \in \mathcal{R}} (-1)^{|c|} + \sum_{c \in \mathcal{M}} (-1)^{|c|} + \sum_{c \in \mathcal{G}} (-1)^{|c|}.$$

We start by showing that the Hall sum for the set  $\mathcal{M}$  is zero.

Let  $c$  be a chain in  $\mathcal{M}$ ,  $\psi_c = \bar{\alpha} \odot_{i,j} \eta$ , and  $\phi_c = \alpha \odot_{i,j} \tau$ .

Define a function  $\Phi$  as follows:

$$\Phi(c) = \begin{cases} c \setminus \{\alpha \odot_{i,j} \eta\} & \text{if } \eta = \tau, \\ c \cup \{\alpha \odot_{i,j} \eta\} & \text{otherwise.} \end{cases}$$

We have two cases to consider. Either  $\eta = \tau$ , or  $\eta \neq \tau$ .

Case 1:  $\eta = \tau$ .

The chain  $c$  has a segment  $\bar{\alpha} \odot_{i,j} \eta < \alpha \odot_{i,j} \eta$ . If  $\eta = \beta$ , then  $\psi_c = \kappa_c$ . Further,  $\kappa_c$  is a proper reduction since  $\eta$  is minimal, so  $c \in \mathcal{R}$ , thus we must have  $\eta < \beta$ , and so  $\Phi(c) = c'$  is a chain.

Now, since  $\eta \neq \beta$ , it follows that  $\Phi(c)$  contains a segment  $\bar{\alpha} \odot_{i,j} \eta < \alpha \odot_{i,j} \zeta$  for some  $\zeta \leq \beta$ . Since  $\eta < \beta$ ,  $\bar{\alpha} \odot_{i,j} \eta$  is not a reduction. If  $\psi_{c'} = \kappa_{c'}$ , then  $c' \notin \mathcal{R}$ . If  $\psi_{c'} \neq \kappa_{c'}$ , then we must have  $\kappa_{c'} = \kappa_c$ , and so again  $c' \notin \mathcal{R}$ . Since we have  $\psi_{c'} = \psi_c$ , it then follows that  $c \in \mathcal{M}$ .

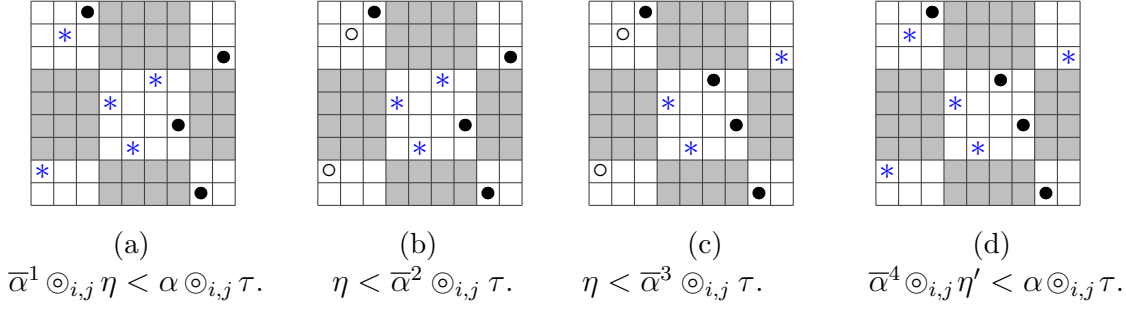


Figure 7.4: Examples to illustrate the steps in case 2. The left-hand side of the inequality is shown as  $*$ .

Case 2:  $\eta \neq \tau$ .

The chain  $c$  has a segment  $\bar{\alpha} \odot_{i,j} \eta < \alpha \odot_{i,j} \tau$ . Clearly,  $\bar{\alpha} \odot_{i,j} \eta < \alpha \odot_{i,j} \eta$ . so  $c' = \Phi(c)$  can fail to be a chain if and only if  $\alpha \odot_{i,j} \eta \not\leq \alpha \odot_{i,j} \tau$  or equivalently, from the nesting condition,  $\eta \not\leq \tau$ .

We show that  $\eta < \tau$  by assuming otherwise, and showing that this leads to a contradiction. To aid the reader, we provide a running example using a generalised balloon, where  $\alpha = 24513$ ,  $\tau = 3142$ ,  $\psi_c = 15324$ , and  $\psi_c$  is matryoshka, with representation  $\bar{\alpha} \odot_{i,j} 213$ . Of course, in this example the representation of  $\psi_c$  is not minimal, and in addition  $\eta < \tau$ , but despite this we feel that the diagrams are a helpful aid in understanding the steps in our argument.

The chain  $c$  contains a segment  $\bar{\alpha}^1 \odot_{i,j} \eta < \alpha \odot_{i,j} \tau$ . Figure 7.4(a) shows how  $\psi_c = \bar{\alpha}^1 \odot_{i,j} \eta$  might be embedded in  $\phi_c = \alpha \odot_{i,j} \tau$ .

Since  $\bar{\alpha}^1 \odot_{i,j} \eta < \alpha \odot_{i,j} \tau$ , the points of  $\bar{\alpha}^1$  in  $\bar{\alpha}^1 \odot_{i,j} \eta$  are a proper subset of the points of  $\alpha$  in  $\alpha \odot_{i,j} \tau$ . We can remove the points of  $\bar{\alpha}^1$  from both sides of the inequality and we obtain  $\eta < \bar{\alpha}^2 \odot_{i,j} \tau$ . This is shown in Figure 7.4(b).

Since, by assumption,  $\eta \not\leq \tau$ , it follows that we must be able to find an embedding of  $\eta$  in  $\bar{\alpha}^2 \odot_{i,j} \tau$  that uses at least one point that is not in  $\tau$ , and so we can write  $\eta < \bar{\alpha}^3 \odot_{i,j} \tau$ , where the points in  $\tau$  that are used is a proper subset of the points of  $\tau$  that are used in  $\eta < \bar{\alpha}^2 \odot_{i,j} \tau$ . An example is shown in Figure 7.4(c).

Now note that the points from  $\alpha$  in this new embedding of  $\eta$ ,  $\bar{\alpha}^3$ , must be disjoint

from the points from  $\alpha$  in the original embedding,  $\bar{\alpha}^1$ . It follows that we can write  $\bar{\alpha}^1 \odot_{i,j} \eta$  as  $\bar{\alpha}^4 \odot_{i,j} \eta'$ , where the points in  $\bar{\alpha}^4$  are the union of the points in  $\bar{\alpha}^1$  and  $\bar{\alpha}^3$ . Further, we must have  $|\eta'| < |\eta|$ . This is shown in Figure 7.4(d).

We now have  $\psi_c = \bar{\alpha}^4 \odot_{i,j} \eta'$ , where  $|\eta'| < |\eta|$ , but this is a contradiction, since  $\bar{\alpha}^1 \odot_{i,j} \eta$  is matryoshka, and so  $\eta$  is minimal. Therefore our assumption must be wrong, and so we have  $\eta < \tau$ , and thus  $c' = \Phi(c)$  is a chain.

It remains to show that  $c' \in \mathcal{M}$ . Now,  $c'$  has a segment  $\bar{\alpha} \odot_{i,j} \eta < \alpha \odot_{i,j} \eta < \alpha \odot_{i,j} \tau$ . Since  $\psi_c = \psi_{c'}$ , and  $\psi_c$  is matryoshka, the only way for  $c' \notin \mathcal{M}$  is if  $\kappa_{c'}$  is a proper reduction.

Since  $\psi_{c'}$  is the pivot of  $c'$ , and there are at least two permutations above  $\psi_{c'}$  in  $c'$ , it follows that  $\kappa_{c'}$  is a complete permutation, and such a permutation cannot be a proper reduction. It follows, therefore, that  $c' \notin \mathcal{R}$ . Since  $\psi_{c'} = \psi_c$ , this then means that  $c' \in \mathcal{M}$ .

We now have that  $\Phi(c)$  is a parity-reversing involution on  $\mathcal{M}$ , and so, by Corollary 1, we have  $\sum_{c \in \mathcal{M}} (-1)^{|c|} = 0$ .

We now have

$$\mu[\pi] = \sum_{c \in \mathcal{R}} (-1)^{|c|} + \sum_{c \in \mathcal{G}} (-1)^{|c|}.$$

The chains in  $\mathcal{R}$  are characterised by the second-highest element being an element of  $\mathcal{R}_\pi$ . Using Corollary 56 from Chapter 6 on the sum over chains in  $\mathcal{R}$  completes the proof.  $\square$

Theorem 69 is hard to use in practice, because of the second term,  $\sum_{c \in \mathcal{G}} (-1)^{|c|}$ .

There is some numerical evidence, based on analysing permutations with length 12 or less, that this second term is, in fact, zero in many cases.

Clearly, one way to handle the difficulties of this second term would be to ensure that the sum was zero. The easiest case is where  $\mathcal{G} = \emptyset$ . This occurs when every permutation in the poset is matryoshka. We state this formally as



**Corollary 70** (to Theorem 69). *If  $\pi = \alpha \odot_{i,j} \beta$ , and every permutation in  $[1, \pi)$  is matryoshka, then*

$$\mu[\pi] = - \sum_{\lambda \in \mathbf{R}_\pi} \mu[\lambda].$$

*Proof.* If every permutation in  $[1, \pi)$  is matryoshka, then  $\mathcal{G} = \emptyset$ , and the result follows immediately.  $\square$

It is easy to see that if  $\pi$  is a direct or skew sum, then every permutation contained in  $\pi$  is matryoshka. In the following section we show that if  $\pi$  is a wedge permutation, then every permutation contained in  $\pi$  is matryoshka.

## 7.5 The principal Möbius function of wedge permutations

We will start by showing:

**Theorem 71.** *If  $\pi$  is a wedge permutation, as defined in Sub-section 7.3.2, then every permutation in  $[1, \pi)$  is matryoshka.*

*Proof.* To prove Theorem 71, it is sufficient to show that an arbitrary permutation contained in a wedge permutation is matryoshka.

Let  $\pi = \alpha \triangle_k \beta$ , and let  $\sigma < \pi$ . Consider any embedding of  $\sigma$  in  $\pi$ . Then if there are  $n$  blue points in the embedding, these, from the construction method, will represent the top  $n$  points of the permutation  $\sigma$ .

Now assume we have two embeddings of  $\sigma$ , say  $\bar{\alpha}^1 \odot_{i,j} \eta$  and  $\bar{\alpha}^2 \odot_{i,j} \zeta$ . If  $|\eta| = |\zeta|$ , then we have  $\eta = \zeta$ . Assume now, without loss of generality, that  $|\eta| < |\zeta|$ . Then  $\eta$  is contained in  $\zeta$ , as the top  $|\eta|$  points of  $\zeta$  are order-isomorphic to  $\eta$ .

For any  $\sigma$  there will be some  $\eta$  that is minimal, noting that this may mean that  $\eta = \epsilon$ .

This then means that any permutation contained in a wedge permutation is matryoshka.  $\square$

We now have

**Lemma 72.** *If  $\pi = \alpha \triangle_k \beta$ , then*

$$\mu[\pi] = - \sum_{\lambda \in R_\pi} \mu[\lambda].$$

*Proof.* By Theorem 71 every permutation in a wedge permutation is matryoshka. Applying Corollary 70 then gives us the result.  $\square$

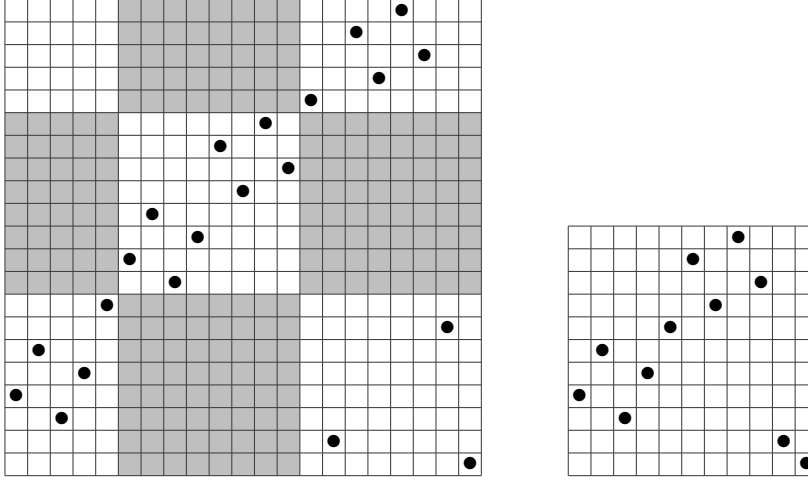


Figure 7.5: The permutations  $\pi$  and  $\sigma$  that demonstrate that  $\sigma$  is defective in the interval  $[1, \pi]$ .

We have shown that every permutation in a wedge permutation is matryoshka. This is not the case for balloon permutations in general. If

$$\alpha = 4, 6, 3, 5, 8, 9, 2, 12, 10, 13, 11, 7, 1$$

$$\beta = 2, 4, 1, 3, 7, 5, 8, 6$$

$$\pi = \alpha \odot_{5,8} \beta$$

$$= 4, 6, 3, 5, 8, 10, 12, 9, 11, 15, 13, 16, 14, 17, 2, 20, 18, 21, 19, 7, 1$$

and

$$\sigma = (2413 \oplus 1 \oplus 3142) \ominus 21$$

$$= 4, 6, 3, 5, 7, 10, 8, 11, 9, 2, 1$$

as shown in Figure 7.5, then  $\sigma$  is not complete, as  $\alpha$  has 13 points, but  $\sigma$  only has 11. Further,  $\sigma$  is not a proper reduction, as it does not contain an interval copy of  $\beta$ . Finally,  $E_{\sigma(\pi)} = \{2413, 3142, 24135, 13524\}$  contains two permutations of length 4, 2413 and 3142, and no permutations with length less than 4. It follows that  $\sigma$  is not matryoshka, and therefore  $\sigma$  must be defective.

We do not claim that  $\pi$  and  $\sigma$  above are minimal, although a (fairly restricted)

computer search failed to find any smaller counter-examples.

We now show that

**Theorem 73.** *If  $\pi = \alpha \triangle_k \beta$ , then  $\mu[\pi] = c \cdot \mu[\beta]$ , where  $c$  is an integer.*

*Proof of Theorem 73.* Our proof will use Lemmas 51 and 52 from Chapter 6, which we restate here for convenience.

**Lemma 74** (Lemma 51 from Chapter 6). *If  $\pi$  has a long corner, then  $\mu[\pi] = 0$ .*

**Lemma 75** (Lemma 52 from Chapter 6). *If  $\pi$  can be written as  $\pi = 1 \oplus \tau$ , or  $\pi = \tau \oplus 1$  or  $\pi = 1 \ominus \tau$  or  $\pi = \tau \ominus 1$ , and does not have a long corner, then  $\mu[\pi] = -\mu[\tau]$ .*

Let  $\pi = \alpha \triangle_k \beta$ .

First, consider the case when  $|\alpha| = 1$ , so either  $\pi = 1 \oplus \beta$  or  $\pi = \beta \ominus 1$ . In the first sub-case, if  $\beta$  begins 1, then  $\rho$  has a long corner, and so by Lemma 74  $\mu[\pi] = 0$ . If  $\beta$  does not begin 1, then by Lemma 75  $\mu[\pi] = -\mu[\beta]$ . The argument for the second sub-case is similar. Thus we have that Theorem 73 is true if  $|\alpha| = 1$ .

Now assume that Theorem 73 is true for all  $\alpha$  with  $|\alpha| < m$ , for some  $m \geq 1$ , and assume that  $|\alpha| = m$ . Let  $\rho$  be a reduction of  $\pi$ . From the definition of a reduction we have that  $\rho = \alpha' \triangle_\ell \beta$  or  $\rho = \beta$ , where  $|\alpha'| < |\alpha|$ . In the first case, by the inductive hypothesis, we have that  $\mu[\rho]$  is a multiple of  $\mu[\beta]$ . In the second case, trivially,  $\mu[\rho] = \mu[\beta]$ . Now summing over all reductions, we can see that  $\mu[\pi] = c \cdot \mu[\beta]$  as required.  $\square$

## 7.6 Chapter summary

The main results from this chapter are expressions for the principal Möbius function of balloon permutations, and a somewhat simpler expression for the principal Möbius function of wedge permutations. We also have that if  $\pi$  is a wedge permutation that can be written as  $\alpha \triangle_k \beta$ , then  $\mu[\pi]$  is a multiple of  $\mu[\beta]$ .

We have observed that if we repeatedly iterate the wedge construction, that is, we define

$$\begin{aligned} W_{k,\alpha,\beta,0} &= \beta, \\ W_{k,\alpha,\beta,1} &= \alpha \triangle_k \beta \\ W_{k,\alpha,\beta,n} &= \alpha \triangle_k (W_{k,\alpha,\beta,n-1}) \text{ for } n > 1 \end{aligned}$$

then the sequence given by

$$\mu[W_{k,\alpha,\beta,0}], \mu[W_{k,\alpha,\beta,1}], \mu[W_{k,\alpha,\beta,2}], \mu[W_{k,\alpha,\beta,3}], \dots$$

follows one of the following patterns:

$0, 0, 0, 0, 0, \dots$	$c, 0, 0, 0, 0, \dots$	$c, c, 0, 0, 0, \dots$
$c, -c, 0, 0, 0, \dots$	$c, 2c, 4c, 8c, 16c, \dots$	$c, c, 2c, 4c, 8c, \dots$
$c, -c, c, -c, c, \dots$	$c, c, c, c, c, \dots$	$c, d, d, d, d, \dots$

where  $c = \mu[\beta]$ , and, where appropriate,  $d = \mu[\alpha \triangle_k \beta]$ .

For some examples, we have outline proofs that these patterns will continue, based on a specific analysis of the proper reductions. However, we do not yet have a way to characterise wedge permutations in general, so that we can predict their behaviour without conducting a complete analysis of the proper reductions. We remark that this is one of the aspects that we are still researching.

# Chapter 8

## Conclusion

### 8.1 A review of our results

Our journey through the Möbius function on the permutation pattern poset started, in Chapter 4, by looking at ways in which we could calculate the value of the Möbius function in a more efficient way than using the recursive definition of Equation 2.1, and here we found that we could reduce the number of permutations that needed to be considered slightly in the general case, and by a very significant number in the case of increasing oscillations.

Chapter 5 continues this theme by showing that if a permutation has opposing adjacencies, then the value of the principal Möbius function is zero. We also describe other cases where, if  $\sigma$  meets certain criteria, then any permutation  $\pi$  that contains  $\sigma$  as an interval has  $\mu[\pi] = 0$ . The main result from this chapter is, however, not the results that provide a way to determine the value of the Möbius function, but the result that, asymptotically, 39.95% of permutations are Möbius zeros. This represents a move away from finding ways to determine the value of the Möbius function towards ways to better understand the permutation pattern poset.

Chapter 6 finds a recursion for the value of the principal Möbius function of 2413-balloons, and uses this to show that  $\text{AbsMax}_\mu(n)$  grows at least exponentially. In this chapter the main result is the exponential growth, and to a large extent the results

for the values of the principal Möbius function of 2413-balloons is the mechanism we use to prove it.

Our results from Chapter 7 show that for balloon permutations generally the value of the principal Möbius function is, essentially, related to the permutations  $\beta$  plus a correction factor. For wedge permutations, this correction factor is guaranteed to be zero.

If we consider the results directly relating to the Möbius function from Chapter 5, one aspect that could be used to distinguish them is that they are all related to finding a set of permutations  $S$  with the property that if a larger permutation contains each permutation in  $S$  as an interval copy, then the value of the Möbius function is zero.

Now compare this with Chapter 6. Given some permutation  $\pi = 2413 \odot \beta$ , first note that  $\beta$  is an interval in  $\pi$ , and so we can (somewhat trivially) claim that  $\pi$  contains an interval copy of  $\beta$ . Our results, with some small exceptions, all give the value of  $\mu[\pi]$  as a multiple of  $\mu[\beta]$ .

We also note that Conjectures 66 and 67 relate to permutations  $\pi = 2413 \odot_{i,j} \beta$ , and again here we see that  $\beta$  occurs as an interval copy in  $\pi$ .

Finally, if we look at the results from Chapter 7, again we can see that  $\beta$  occurs as an interval copy in  $\alpha \odot_{i,j} \beta$  and, trivially, in  $\alpha \triangle_k \beta$ .

Our suggestion here is that, in some ill-defined sense, intervals in permutations have a marked effect on the value of the principal Möbius function. In some cases, the presence of a copy interval or intervals guarantees that the value of the principal Möbius function is zero. In other cases, where we have an interval copy of  $\beta$ , we see that the value of the principal Möbius function can be expressed in terms of  $\mu[\beta]$ , with, possibly, some correction factor, although for 2413-balloons and wedge permutations, the correction factor is zero for all but trivial cases.

## 8.2 Further research into the Möbius function

In the chapter summaries we have already mentioned several possible avenues for future research. We now consider more general avenues for further research, and we divide these into two main areas.

The first area is research that will give expressions or recursions for the value of the Möbius function on some interval. Current results, and indeed our active research, tends to examine permutations that have some “structure”. We do not intend to try and formally define structure; rather we claim that it is a property that is generally recognisable when it is seen. The classic example of a set of permutations with structure is, we suggest, the decomposable and separable permutations, studied by Burstein, Jelínek, Jelínková and Steingrímsson [18], as, in some ill-defined sense, these permutations have a lot of structure. Our own research into permutations with opposing adjacencies, and permutations that are 2413-balloons again looks at permutations with structure.

The second area is research into what we call “global” properties of the permutation pattern poset. The result from Chapter 5 that 39.95% of permutations are Möbius zeros is one example, as is the exponential growth rate of the principal Möbius function proved in Chapter 6.

### 8.2.1 The Möbius function of simple permutations

Recall that the simple permutations are permutations that only contain trivial intervals. In other areas of permutation patterns, such as the enumeration of permutation classes, the simple permutations underpin many results. For further details, we refer the reader to the survey article by Brignall [13].

By contrast, little is known about the principal Möbius function of simple permutations. The first reference to this area occurs in the concluding remarks of Burstein, Jelínek, Jelínková and Steingrímsson [18]. Here (using the terminology of this thesis) they give a sequence of values of  $\text{AbsMax}_\mu(n)$  for  $n = 1, \dots, 11$ , and they



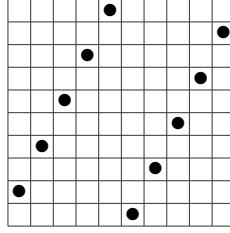


Figure 8.1: A simple parallel alternation.

note that there is, up to symmetry, a unique permutation  $\pi_n$  of length  $n$  such that  $|\mu[\pi_n]| = \text{AbsMax}_\mu(n)$ . They further note that  $\pi_n$  is simple except for the case  $n = 3$ , (but there are no simple permutations of length 3). The author has confirmed that this is also the case for permutations of length 12 and 13 (see Table 8.1 on page 150).

To understand the principal Möbius function of simple permutations, we begin by considering some examples that are easy to describe, as they have recognisable structure.

A simple parallel alternation is a permutation  $\pi$  with even length  $2n$ , where  $\pi = 2, 4, \dots, 2n, 1, 3, \dots, 2n - 1$ , or any symmetry of this sequence. We show an example of a simple parallel alternation in Figure 8.1. Smith's paper on permutations with one descent [44] covers simple parallel alternations, and so we have an explicit expression for their principal Möbius function value. If  $\pi$  is a simple parallel alternation of length  $n$ , then

$$\mu[\pi] = -\binom{\frac{n}{2} + 1}{2}.$$

Increasing oscillations are also simple permutations. Our paper on permutations with an indecomposable lower bound [16], on which Chapter 4 is based, includes a recursion for the principal Möbius function of increasing oscillations. We are unaware of any other published results for families of simple permutations.

We now provide several examples of simple permutations with a recognisable structure, and give conjectures for the value of the principal Möbius function.

We have already described simple parallel alternations. We extend our vocabulary to define an *alternation* to be a permutation where every odd entry is to the right

of every even entry, or a symmetry of such a permutation. A *wedge alternation* is then an alternation where the two sets of entries point in opposite directions, and an example is shown in Figure 8.2. Wedge alternations are not simple, but a single point can be added in one of two ways to form a simple permutation. These are called type 1 and type 2 wedge simples, written  $W_1(n)$  and  $W_2(n)$ , where we require  $n > 3$ . The wedge simples appear to have been introduced in [17], where we have

**Theorem 76** (Brignall, Huczynska and Vatter [17, Theorem 3]). *For any fixed  $k$ , every sufficiently long simple permutation contains either a proper pin sequence of length at least  $k$ , a parallel alternation of length at least  $k$ , or a wedge simple permutation of length at least  $k$ .*

We refer the interested reader to [17] for a definition of “proper pin sequence”.

The type 1 and type 2 wedge simples have the form

$$W_1(n) = \begin{cases} 3, 5, \dots, n-1, 1, n, n-2, \dots, 2 & \text{If } n \text{ is even,} \\ 3, 5, \dots, n, 1, n-1, n-3, \dots, 2 & \text{If } n \text{ is odd,} \end{cases}$$

and

$$W_2(n) = \begin{cases} 2, 4, \dots, n-2, n, n-3, n-5, \dots, 1, n-1 & \text{if } n \text{ is even,} \\ 2, 4, \dots, n-3, n, n-2, n-4, \dots, 1, n-1 & \text{if } n \text{ is odd,} \end{cases}$$

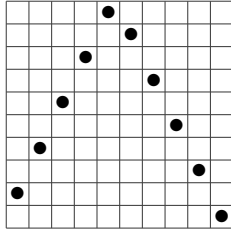
and every symmetry of these permutations. Examples of type 1 and type 2 wedge simples are shown in Figure 8.2. We calculated the value of the principal Möbius function of  $W_1(n)$  and  $W_2(n)$  for  $n = 4, \dots, 30$ , and the results are shown in Tables A.1 and A.2 in Appendix A.

These values suggest the following conjecture:

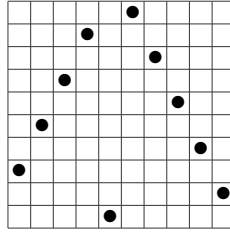
**Conjecture 77.** For all  $n > 3$ ,

$$\mu[W_1(n)] = (-1)^n 3(3-n)$$

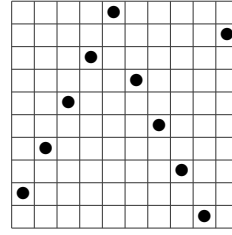
$$\mu[W_2(n)] = (-1)^n (1-n).$$



A wedge alternation.



A type 1 wedge simple



A type 2 wedge simple

Figure 8.2: Examples of a wedge alternation, and type 1 and type 2 wedge simple permutations.

Schmerl and Trotter [41] show that every simple permutation of length  $n$  contains a simple permutation of length  $n - 1$  or  $n - 2$ . Further, they show that the only *exceptional* simple permutations, which are those permutations of length  $n$  that do not contain a simple sub permutation of length  $n - 1$ , are the parallel alternations, which we discussed above. The *nearly-exceptional* simple permutations are permutations which are not exceptional, but there is only a single point which, when deleted, results in a smaller simple permutation; so deleting any other point results in a non-simple permutation. The nearly-exceptional permutations have not, as far as we are aware, appeared in any publication. They were described in a talk given by Robert Brignall at Permutation Patterns 2010 [12]. There are three types of nearly exceptional simple permutations, which we refer to as  $E_1(2n, k)$ ,  $E_2(2n)$  and  $O(2n + 1, k)$ . The first parameter gives the length of the permutation, which, for simplicity, we require to be greater than 5. For  $E_1(2n, k)$ , the second parameter,  $k$  must satisfy  $1 \leq k \leq n - 2$ , and for  $O(2n + 1, k)$ ,  $k$  must satisfy  $1 \leq k \leq n - 1$ . Formally, up to symmetry, we have

$$\begin{aligned} E_1(2n, k) = & n + 1, 1, n + 2, 2, \dots, \\ & k, n + k + 1, n, n + k + 2, n - 1, \dots, \\ & 2n, k + 1; \end{aligned}$$

$$E_2(2n) = n, 1, n + 1, 2, \dots, 2n - 2, 2n, n - 1, 2n - 1;$$

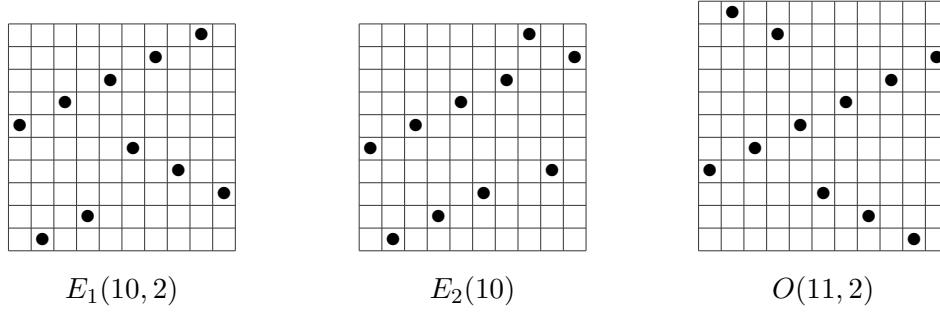


Figure 8.3: Examples of nearly exceptional simple permutations.

and

$$\begin{aligned}
 O(2n+1, k) = & n-k+1, 2n+1, n-k+2, 2n, \dots, \\
 & 2n+2-k, n+1, n-k, n+2, n-k-1, \dots, \\
 & 1, 2n-k+1.
 \end{aligned}$$

Examples of  $E_1(2n, k)$ ,  $E_2(2n)$  and  $O(2n+1, k)$  are shown in Figure 8.3. We calculated the value of the principal Möbius function of  $E_1(2n, k)$ ,  $E_2(2n)$ , and  $O(2n+1, k)$ , for  $n = 3, \dots, 15$ , and all valid values of  $k$ , and the results are shown in Tables A.3, A.4, and A.5 in Appendix A.

These values suggest the following conjectures:

**Conjecture 78.** For all  $n \geq 5$  and for all  $k$  with  $1 \leq k \leq n-2$ ,

$$\mu[E_1(2n, k)] = -\frac{(4n-2) + k^2 - k}{2}.$$

**Conjecture 79.** For all  $n \geq 5$ ,

$$\mu[E_2(2n)] = \frac{n - n^2 - 4}{2}.$$

**Conjecture 80.** For all  $n \geq 5$  and for all  $k$  with  $1 \leq k \leq n-1$ ,

$$\mu[O(2n+1, k)] = 2n.$$

We remark that these permutations do exhibit a significant amount of structure, and we suggest that finding an expression for the principal Möbius function of other, less-structured, simple permutations will be difficult.

## 8.3 Further research into global properties of the poset

The second area for future research is to examine global attributes of the permutation pattern poset. Our results for the proportion of permutations that are Möbius zeros in Chapter 5 based on [15], and our result on the growth of  $\text{AbsMax}_\mu(n)$  in Chapter 6 based on [31] are examples of existing results.

### 8.3.1 Extremal values of $\mu[\pi]$ as a function of the length of the permutations

We think that further examination of the behaviour of  $\text{AbsMax}_\mu(n)$  is one interesting area for research. We could also define  $\text{Max}_\mu(n) = \max\{\mu[\pi] : |\pi| = n\}$ , and  $\text{Min}_\mu(n) = \min\{\mu[\pi] : |\pi| = n\}$ , and then ask how these two functions behave as functions of  $n$ . Trivially, we have that, for  $n > 4$ ,  $\text{Max}_\mu(n+1) \geq -\text{Min}_\mu(n)$  and  $\text{Min}_\mu(n+1) \leq -\text{Max}_\mu(n)$ . Table 8.1 shows the first thirteen values of these functions.

### 8.3.2 Characterising permutations where $\mu[\pi]$ achieves extreme values

The functions discussed above,  $\text{AbsMax}_\mu(n)$ ,  $\text{Max}_\mu(n)$ , and  $\text{Min}_\mu(n)$ , concern themselves with the minimum or maximum values of the principal Möbius function. We could also ask for the characteristics of the permutations that achieve the minimum or maximum values. This question was, of course, first asked implicitly in Burstein, Jelínek, Jelínková and Steingrímsson [18], where there is a remark that up to length 11, if  $\pi$  is a permutation where  $|\mu[\pi]| = \text{AbsMax}_\mu(n)$ , then  $\pi$  is simple, and, up to symmetry, unique. We remark here that our own intuition is that if  $|\mu[\pi]| = \text{AbsMax}_\mu(n)$ , then  $\pi$  is likely to be simple, but we suspect, for sufficiently

$n$	$\text{Min}_\mu(n)$	$\text{Max}_\mu(n)$
1	1	1
2	-1	-1
3	0	1
4	-3	0
5	0	6
6	-11	1
7	-2	15
8	-27	14
9	-50	39
10	-58	55
11	-81	143
12	-261	183
13	-330	261

Table 8.1: Values of  $\text{Max}_\mu(n)$  and  $\text{Min}_\mu(n)$  for  $n = 1, \dots, 13$ .

large lengths, there may be more than one canonical permutation that meets the equality.

A slightly different approach would be to consider the set of canonical permutations that achieve  $\text{Max}_\mu(n)$  or  $\text{Min}_\mu(n)$ . We have results up to length 13, and these are shown in Table B.1 in Appendix B.

We remark that, as with many aspects of the permutation pattern poset, results for small permutations are atypical, and, in our opinion, the entries in the table for  $n \leq 7$  certainly fall into the atypical category.

We can see that  $\text{Max}_\mu(13)$  is attained by three permutations. If we set

$$\theta = 4, 7, 2, 10, 5, 1, 12, 8, 3, 11, 6, 9$$

then the first two permutations listed achieving  $\text{Max}_\mu(13)$  are  $1 \oplus \theta$ , and  $1 \oplus ((\theta)^R)^{-1}$  respectively. With this exception, all of the permutations that achieve  $\text{Min}_\mu(n)$  or  $\text{Max}_\mu(n)$  for  $n > 7$  are simple.

We suggest that, for lengths greater than 7, the permutations that achieve  $\text{Max}_\mu(n)$

are either simple, or they are a sum (either direct or skew) of 1 and a permutation that achieves  $\text{Min}_\mu(n - 1)$ . Conversely, the permutations that achieve  $\text{Min}_\mu(n)$  are either simple, or they are a sum of 1 and a permutation that achieves  $\text{Max}_\mu(n - 1)$ .





# Appendix A

## Values of $\mu[\pi]$ where $\pi$ is a specific simple permutation

### A.1 Values of $\mu[W_1(n)]$

$n$	$\mu[W_1(n)]$	$n$	$\mu[W_1(n)]$	$n$	$\mu[W_1(n)]$
4	-3	13	30	22	-57
5	6	14	-33	23	60
6	-9	15	36	24	-63
7	12	16	-39	25	66
8	-15	17	42	26	-69
9	18	18	-45	27	72
10	-21	19	48	28	-75
11	24	20	-51	29	78
12	-27	21	54	30	-81

Table A.1: Values of  $\mu[W_1(n)]$  for  $n = 4, \dots, 30$ .

## A.2 Values of $\mu[W_2(n)]$

$n$	$\mu[W_2(n)]$	$n$	$\mu[W_2(n)]$	$n$	$\mu[W_2(n)]$
4	-3	13	12	22	-21
5	4	14	-13	23	22
6	-5	15	14	24	-23
7	6	16	-15	25	24
8	-7	17	16	26	-25
9	8	18	-17	27	26
10	-9	19	18	28	-27
11	10	20	-19	29	28
12	-11	21	20	30	-29

Table A.2: Values of  $\mu[W_2(n)]$  for  $n = 4, \dots, 30$ .

### A.3 Values of $\mu[E_1(2n, k)]$

$n$	$k$	$\mu[E_1(2n, k)]$	$n$	$k$	$\mu[E_1(2n, k)]$	$n$	$k$	$\mu[E_1(2n, k)]$
3	1	-5	8	1	-25	11	1	-21
			8	2	-16	11	2	-22
4	1	-7	8	3	-18	11	3	-24
4	2	-8	8	4	-21	11	4	-27
			8	5	-25	11	5	-31
5	1	-9	8	6	-30	11	6	-36
5	2	-10				11	7	-42
5	3	-12	9	1	-17	11	8	-49
			9	2	-18	11	9	-57
6	1	-11	9	3	-20			
6	2	-12	9	4	-23	12	1	-23
6	3	-14	9	5	-27	12	2	-24
6	4	-17	9	6	-32	12	3	-26
			9	7	-38	12	4	-29
7	1	-13				12	5	-33
7	2	-14	10	1	-19	12	6	-38
7	3	-16	10	2	-20	12	7	-44
7	4	-19	10	3	-22	12	8	-51
7	5	-23	10	4	-25	12	9	-59
			10	5	-29	12	10	-68
			10	6	-34			
			10	7	-40			
			10	8	-47			

Table A.3a: Values of  $\mu[E_1(2n, k)]$  for  $n = 3, \dots, 12$ , and all valid values of  $k$ .

$n$	$k$	$\mu[E_1(2n, k)]$	$n$	$k$	$\mu[E_1(2n, k)]$	$n$	$k$	$\mu[E_1(2n, k)]$
13	1	-25	14	1	-27	15	1	-29
13	2	-26	14	2	-28	15	2	-30
13	3	-28	14	3	-30	15	3	-32
13	4	-31	14	4	-33	15	4	-35
13	5	-35	14	5	-37	15	5	-39
13	6	-40	14	6	-42	15	6	-44
13	7	-46	14	7	-48	15	7	-50
13	8	-53	14	8	-55	15	8	-57
13	9	-61	14	9	-63	15	9	-65
13	10	-70	14	10	-72	15	10	-74
13	11	-80	14	11	-82	15	11	-84
			14	12	-93	15	12	-95
						15	13	-107

Table A.3b: Values of  $\mu[E_1(2n, k)]$  for  $n = 13, 14, 15$ , and all valid values of  $k$ .

## A.4 Values of $\mu[E_2(2n)]$

$n$	$\mu[E_2(2n)]$	$n$	$\mu[E_2(2n)]$	$n$	$\mu[E_2(2n)]$
3	-5	8	-30	13	-80
4	-8	9	-38	14	-93
5	-12	10	-47	15	-107
6	-17	11	-57		
7	-23	12	-68		

Table A.4: Values of  $\mu[E_2(2n)]$  for  $n = 3, \dots, 15$ .

## A.5 Values of $\mu[O(2n + 1, k)]$

$n$	$k$	$\mu[O(2n + 1, k)]$	$n$	$k$	$\mu[O(2n + 1, k)]$
3	1, 2	6	10	1, ..., 9	20
4	1, 2, 3	8	11	1, ..., 10	22
5	1, ..., 4	10	12	1, ..., 11	24
6	1, ..., 5	12	13	1, ..., 12	26
7	1, ..., 6	14	14	1, ..., 13	28
8	1, ..., 7	16	15	1, ..., 14	30
9	1, ..., 8	18			

Table A.5: Values of  $\mu[O(2n + 1, k)]$  for  $n = 3, \dots, 15$ , and all valid values of  $k$ .

## Appendix B

**Canonical permutations that  
achieve  $\text{Max}_\mu(n)$  or  $\text{Min}_\mu(n)$**



$n$	$\mu[\pi] = \text{Min}_\mu(n)$	$\mu[\pi] = \text{Max}_\mu(n)$
1	<b>1</b>	<b>1</b>
2	<b>12</b>	<b>12</b>
3	123	132
4	<b>2413</b>	1234; 1243; 1432
5	$\left\{ \begin{array}{l} 12345; 12354; 12435; 12453; \\ 12543; 13452; 14325; 14532; \\ 15432; 21354; 21453; 21543 \end{array} \right.$	<b>24153</b>
6	<b>351624</b>	231564
7	2547163; 3416725	<b>2461735</b>
8	<b>35172846</b>	<b>36184725</b>
9	<b>472951836</b>	<b>357182946</b>
10	<b>4,6,8,1,9,2,10,3,5,7</b>	<b>4,7,9,1,10,6,2,8,3,5</b>
11	<b>3,5,8,10,1,7,11,2,9,4,6</b>	<b>3,6,1,9,4,11,7,2,10,5,8</b>
12	<b>4,7,2,10,5,1,12,8,3,11,6,9</b>	<b>5,10,2,7,12,4,9,1,6,11,318</b>
13	$\left\{ \begin{array}{l} \mathbf{6,2,9,4,11,1,7,13,3,10,5,12,8} \\ 1,5,8,3,11,6,2,13,9,4,12,7,10; \\ 1,8,11,5,13,9,3,12,6,2,10,4,7; \\ \mathbf{4,7,2,10,5,13,1,12,8,3,11,6,9} \end{array} \right.$	

Table B.1: Canonical permutations of length  $n$  where the principal Möbius function has a minimum / maximum value. Simple permutations are highlighted.

# Bibliography

- [1] M. H. Albert and M. D. Atkinson. Simple permutations and pattern restricted permutations. *Discrete Mathematics*, 300(1-3):1–15, 2005. Cited on pages [66](#), [92](#), and [103](#).
- [2] M. H. Albert, M. D. Atkinson, and M. Klazar. The enumeration of simple permutations. *Journal of Integer Sequences*, 6(4), 2003. Cited on pages [23](#), [81](#), [88](#), and [92](#).
- [3] M. D. Atkinson and T. Stitt. Restricted permutations and the wreath product. *Discrete Mathematics*, 259(1-3):19–36, 2002. Cited on page [66](#).
- [4] A. Bernini, M. Cervetti, L. Ferrari, and E. Steingrímsson. Enumerative combinatorics of intervals in the Dyck pattern poset. <https://arxiv.org/abs/1910.00299>. Cited on page [23](#).
- [5] A. Bernini and L. Ferrari. Vincular pattern posets and the Möbius function of the quasi-consecutive pattern poset. *Annals of Combinatorics*, 21(4):519–534, 2017. Cited on page [23](#).
- [6] A. Bernini, L. Ferrari, and E. Steingrímsson. The Möbius function of the consecutive pattern poset. *Electronic Journal of Combinatorics*, 18(1):#P146, 2011. Cited on page [22](#).
- [7] A. Björner. The Möbius function of subword order. In *Invariant Theory and Tableaux (Minneapolis, MN, 1988)*, volume 19, pages 118–124. Springer, New York, 1990. Series Title: IMA Volumes in Mathematics and its Applications. Cited on pages [22](#) and [72](#).

- [8] A. Björner. The Möbius function of factor order. *Theoretical Computer Science*, 117(1-2):91–98, 1993. Cited on page 22.
- [9] M. Bóna, editor. *Handbook of Enumerative Combinatorics*. CRC Press, Boca Raton, FL, 2015. Series Title: Discrete Mathematics and its Applications (Boca Raton). Cited on page 18.
- [10] M. Bóna. *Introduction to Enumerative and Analytic Combinatorics*. CRC Press, Boca Raton, FL, second edition, 2016. Series Title: Discrete Mathematics and its Applications (Boca Raton). Cited on page 18.
- [11] P. Brändén and A. Claesson. Mesh patterns and the expansion of permutation statistics as sums of permutation patterns. *Electronic Journal of Combinatorics*, 18(2):#P5, 2011. Cited on page 23.
- [12] R. Brignall. Almost exceptional simple permutations, August 2010. Cited on page 146.
- [13] R. Brignall. A survey of simple permutations. In *Permutation Patterns*, volume 376, pages 41–65. Cambridge University Press, Cambridge, January 2010. Series Title: London Mathematical Society Lecture Note Series. Cited on page 143.
- [14] R. Brignall, V. Jelínek, J. Kynčl, and D. Marchant. Zeros of the Möbius function of permutations. <https://arxiv.org/abs/1810.05449v1>. Cited on page 84.
- [15] R. Brignall, V. Jelínek, J. Kynčl, and D. Marchant. Zeros of the Möbius function of permutations. *Mathematika*, 65(4):1074–1092, 2019. Cited on pages ix, 3, 10, 28, 63, 84, and 149.
- [16] R. Brignall and D. Marchant. The Möbius function of permutations with an indecomposable lower bound. *Discrete Mathematics*, 341(5):1380–1391, 2018. Cited on pages ix, 2, 28, 33, and 144.
- [17] R. Brignall, N. Ruškuc, and V. Vatter. Simple permutations: Decidability and unavoidable substructures. *Theoretical Computer Science*, 391(1-2):150–163, February 2008. Cited on page 145.

- [18] A. Burstein, V. Jelínek, E. Jelínková, and E. Steingrímsson. The Möbius function of separable and decomposable permutations. *Journal of Combinatorial Theory, Series A*, 118(8):2346–2364, 2011. Cited on pages [25](#), [26](#), [35](#), [39](#), [40](#), [57](#), [64](#), [65](#), [72](#), [88](#), [89](#), [98](#), [100](#), [118](#), [119](#), [125](#), [143](#), and [149](#).
- [19] P. J. Cameron. *Combinatorics: Topics, Techniques, Algorithms*. Cambridge University Press, Cambridge, 1994. Cited on page [19](#).
- [20] E. S. Egge and T. Mansour. 231-avoiding involutions and Fibonacci numbers. *Australasian Journal of Combinatorics*, 30:75–84, 2004. Cited on page [14](#).
- [21] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Composito Mathematica*, 2:463–470, 1935. Cited on page [17](#).
- [22] L. Euler. *Institutiones Calculi Differentialis Cum Eius usu in Analysi Finitorum Ac Doctrina Serierum*. Academiae Imperialis Scientiarum Petropolitanae, Saint Petersburg, 1755. Cited on page [17](#).
- [23] L. Euler. Recherches sur une nouvelle espèce des quarrés magiques. *Verhandelingen witgegeven door het zeeuwsch Genootschap der Wetenschappen te Vlissingen*, 9:85–239, 1782. Cited on page [81](#).
- [24] V. Jelínek. Private correspondence, 2017. Cited on pages [57](#) and [62](#).
- [25] V. Jelínek, I. Kantor, J. Kynčl, and M. Tancer. On the growth of the Möbius function of permutations. *Journal of Combinatorial Theory, Series A*, 169:105121, 2020. Cited on pages [28](#), [74](#), [98](#), [112](#), and [121](#).
- [26] I. Kaplansky. The asymptotic distribution of runs of consecutive elements. *The Annals of Mathematical Statistics*, 16(2):200–203, 1945. Cited on page [81](#).
- [27] S. Kitaev. *Patterns in Permutations and Words*. Monographs in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, 2011. Cited on pages [17](#) and [22](#).
- [28] D. E. Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific Journal of Mathematics*, 34(3):709–727, 1970. Cited on page [17](#).

- [29] D. E. Knuth. *The Art of Computer Programming. Vol 1.* Addison-Wesley, Reading, MA., second edition, 1976. Cited on page 17.
- [30] P. A. MacMahon. *Combinatory Analysis, Volume I and II (bound in one volume).* Dover Phoenix Editions. Dover Publications, Inc., Mineola, NY, 2004. Cited on page 17.
- [31] D. Marchant. 2413-balloon permutations and the growth of the Möbius function. *Electronic Journal of Combinatorics*, 27(1):#P1.7, 2020. Cited on pages ix, 5, 29, 97, and 149.
- [32] D. Marchant. Permutation WorkShop. <https://dmarchant.co.uk/pws/pws-downloads.html>, 2020. Cited on page 61.
- [33] P. R. McNamara and E. Steingrímsson. On the topology of the permutation pattern poset. *Journal of Combinatorial Theory, Series A*, 134:1–35, 2015. Cited on pages 26 and 27.
- [34] A. F. Möbius. Über eine besondere art von umkehrung der reihen. *Journal für die Reine und Angewandte Mathematik*, 9:105–123, 1832. Cited on page 19.
- [35] D. Rogers. Ascending sequences in permutations. *Discrete Mathematics*, 22(1):35–40, 1978. Cited on page 17.
- [36] G.-C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte*, 2:340–368, 1964. Cited on pages 19, 20, and 22.
- [37] D. Rotem. Stack sortable permutations. *Discrete Mathematics*, 33(2):185–196, 1981. Cited on page 17.
- [38] M. Rumney and E. J. F. Primrose. A sequence connected with the sub-factorial sequence. *The Mathematical Gazette*, 52(382):381, 1968. Cited on page 81.
- [39] B. E. Sagan and V. Vatter. The Möbius function of a composition poset. *Journal of Algebraic Combinatorics*, 24(2):117–136, 2006. Cited on pages 22, 25, and 72.

- [40] B. E. Sagan and R. Willenbring. Discrete Morse theory and the consecutive pattern poset. *Journal of Algebraic Combinatorics*, 36(4):501–514, 2012. Cited on page [23](#).
- [41] J. H. Schmerl and W. T. Trotter. Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures. *Discrete Mathematics*, 113(1-3):191–205, 1993. Cited on page [146](#).
- [42] R. Simion and F. W. Schmidt. Restricted permutations. *European Journal of Combinatorics*, 6(4):383–406, 1985. Cited on page [17](#).
- [43] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences, published electronically at <https://oeis.org>. Cited on page [81](#).
- [44] J. P. Smith. On the Möbius function of permutations with one descent. *Electronic Journal of Combinatorics*, 21(2):#P2.11, 2014. Cited on pages [27](#), [34](#), [64](#), [98](#), [100](#), [112](#), and [144](#).
- [45] J. P. Smith. Intervals of permutations with a fixed number of descents are shellable. *Discrete Mathematics*, 339(1):118–126, 2016. Cited on pages [27](#) and [72](#).
- [46] J. P. Smith. A formula for the Möbius function of the permutation poset based on a topological decomposition. *Advances in Applied Mathematics*, 91:98–114, 2017. Cited on pages [24](#), [27](#), [28](#), and [72](#).
- [47] J. P. Smith. Private correspondence, 2018. Cited on page [88](#).
- [48] J. P. Smith. On the Möbius function and topology of general pattern posets. *Electronic Journal of Combinatorics*, 26(1):#P1.49, 2019. Cited on page [24](#).
- [49] J. P. Smith and H. Ulfarsson. The poset of mesh patterns. *Discrete Mathematics*, 343(6):111848. Cited on page [23](#).
- [50] R. P. Stanley. *Enumerative Combinatorics, Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012. Cited on pages [12](#), [19](#), [20](#), [22](#), [71](#), [95](#), and [101](#).

- 
- [51] E. Steingrímsson. Some open problems on permutation patterns. In *Surveys in Combinatorics 2013*, volume 409, pages 239–263. Cambridge University Press, Cambridge, 2013. Series Title: London Mathematical Society Lecture Note Series. Cited on page [18](#).
  - [52] E. Steingrímsson and B. E. Tenner. The Möbius function of the permutation pattern poset. *Journal of Combinatorics*, 1(1):39–52, 2010. Cited on page [25](#).
  - [53] M. L. Wachs. Poset topology: Tools and applications. In *Geometric Combinatorics*, volume 13 of *IAS/Park City Mathematics Series*, pages 497–615. American Mathematical Society, Providence, RI, 2007. Cited on page [20](#).
  - [54] H. S. Wilf. The patterns of permutations. *Discrete Mathematics*, 257(2-3):575–583, 2002. Cited on pages [25](#) and [30](#).

# Index

## Symbols

$\mu[\pi]$ , *see* principal Möbius function

$\mu[\sigma, \pi]$ , *see* Möbius function

$\otimes^n \alpha$ , 36

$\oplus^r \alpha$ , 14

$\sigma$ -annihilator, 67

$\sigma$ -core of  $\phi$ , 78

$\sigma$ -narrow, *see* permutation

$\sigma$ -zero, 67

$i$ -switch, *see* embedding

$i, j$ -balloon, 118

2413-balloon, 102

    core, 107

    pivot, 106

    proper reduction, 104

## A

adjacency, 15

adjacency-free, 66

alternation, *see* permutation

annihilator, 67

annihilator multiset, 89

annihilator pair, 67

ascent, 15

avoids, *see* permutation

## B

balloon, *see* permutation

    indexes, 118

    pivot, 131

    proper reduction, 130

basis, *see* permutation class

block diagram, 126

## C

canonical, *see* permutation

chain, 11

    Hall sum, 12

    segment, 13

classic pattern containment, 10

complement, *see* permutation

complete, *see* permutation

complete nearly-layered, *see*  
    permutation

contained in, *see* embedding

contains, *see* permutation

contributing set, 34

contribution, 36

core (2413-balloon), *see* 2413-balloon



core (poset), *see* poset

corners, *see* permutation

## D

defective, *see* permutation

descent, 15

diamond-tipped, *see* poset

direct sum, *see* permutation

double 2413-balloon, 103

down-adjacency, 15

## E

embedding, *see* permutation

*i*-switch, 70

contained in, 70

even, 70

image, 70

odd, 70

sign-reversing, 70

source, 70

even, *see* embedding

exceptional, *see* permutation

## F

face, *see* simplicial complex

family, 38

finest sum decomposition, 13

## H

Hall sum, *see* chain

Hasse diagram, 11

hereditary class, *see* permutation class

## I

image, *see* embedding

improper reduction (of a 2413  
balloon), 104

incidence algebra, 20

zeta function, 21

zeta matrix, 21

incomparable, *see* permutation

increasing oscillating sequence, 37

increasing oscillation, *see* permutation

indexes, *see* balloon

inflation, 66

interleave, *see* permutation

interval, *see* poset, *see* permutation

interval copy, *see* permutation

## L

layered, *see* permutation

least 2413-balloon, 106

locally finite, *see* poset

long corner, 14

## M

Möbius function

$\mu[\sigma, \pi]$ , 11

Möbius zero, 67

matryoshka, *see* permutation

merge, *see* permutation

minimal annihilator multiset, 89

minimal annihilator sets, 89

## N

narrow-tipped, *see* poset

nearly-exceptional, *see* permutation

nesting condition, 125

net contribution, 36

normal embedding, 10

## O

odd, *see* embedding

opposing adjacencies, 15

order-isomorphic, 9

## P

parity-reversing involution, 12

pattern, *see* permutation

permutation, 9

$\sigma$ -narrow, 78

alternation, 144

avoids, 9

balloon, 125

canonical, 16

complement, 15

complete, 129

complete nearly-layered, 59

contains, 9

corners, 100

defective, 131

direct sum, 13

embedding, 10

exceptional, 146

incomparable, 11

increasing oscillation, 37

interleave, 36

interval, 14

interval copy, 14

layered, 14

matryoshka, 131

merge, 125

nearly-exceptional, 146

pattern, 9

proper wedge, 127

reversal, 15

simple, 14

skew interleave, 36

skew sum, 13

skew-decomposable, 13

skew-indecomposable, 13

sum-decomposable, 13

sum-indecomposable, 13

wedge, 127

wedge alternation, 145

permutation class, 15

basis, 15

hereditary class, 15

pivot (in a 2413-balloon), *see*  
2413-balloon

pivot (in a balloon), *see* balloon

poset, 10

core, 68

diamond-tipped, 68

interval, 11

locally finite, 20

narrow-tipped, 68

principal Möbius function

$\mu[\pi]$ , 11

proper reduction (of a 2413-balloon),  
*see* 2413-balloon

proper reduction (of a balloon), *see*  
balloon

proper wedge, *see* permutation

**R**

reduction (of a 2413 balloon), [103](#)

reversal, *see* permutation

**S**

segment, *see* chain

shape, [45](#)

sign-reversing, *see* embedding

simple, *see* permutation

simplicial complex, [19](#)

    face, [19](#)

skew interleave, *see* permutation

skew sum, *see* permutation

skew-decomposable, *see* permutation

skew-indecomposable, *see* permutation

source, *see* embedding

sum-decomposable, *see* permutation

sum-indecomposable, *see* permutation

**T**

triple-adjacency, [15](#)

trivial interval, [14](#)

**U**

up-adjacency, [15](#)

**W**

wedge, *see* permutation

wedge alternation, *see* permutation

**Z**

zeta function, *see* incidence algebra

zeta matrix, *see* incidence algebra